A Tough Nut for Mathematical Knowledge Management

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Motivation

How to query a mathematical database?

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How to query a mathematical database?

Mathematical situation

↓ Formalisation of concepts

Formal situation

↓ Query based on deduction

Answer from the database

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How to query a mathematical database?

Mathematical situation : problem

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Query based on deduction

Answer from the database : theorem

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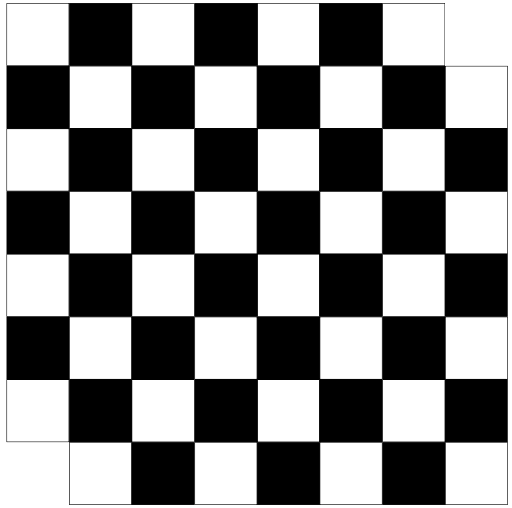
Answer from the database : theorem

Could be useful, *even* for mathematicians, but . . .

The Mutilated Checkerboard Problem



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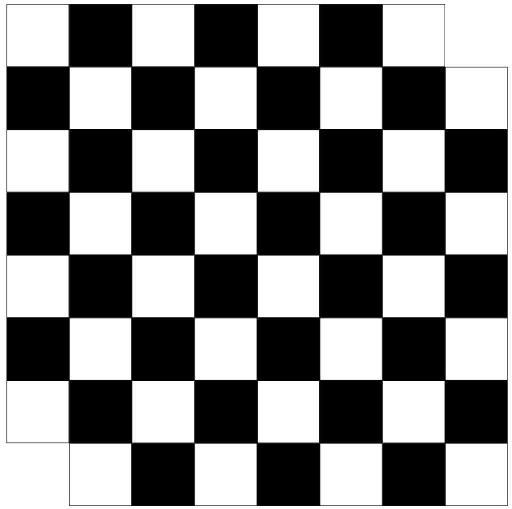

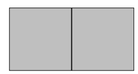

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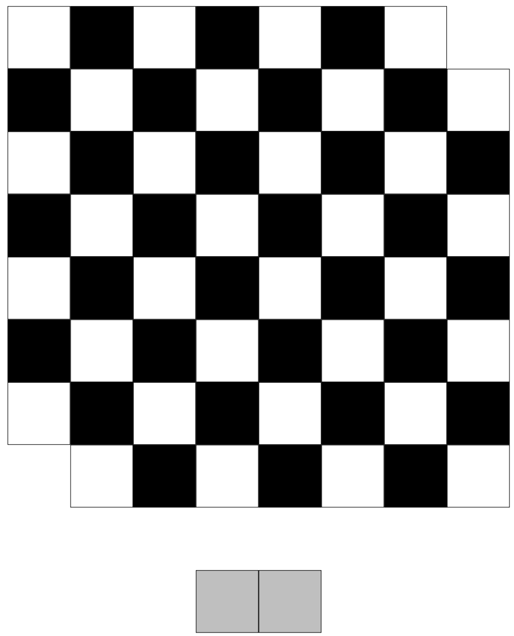
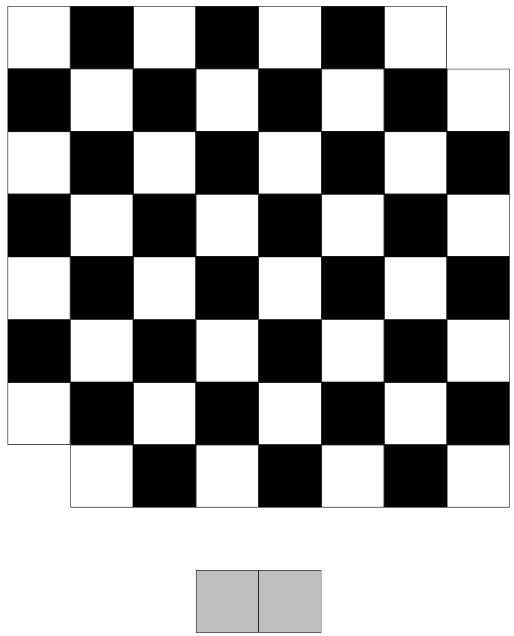


It is impossible to cover the mutilated checkerboard shown in the figure with dominoes like the one in the figure. Namely, a domino covers a square of each color, but there are 30 black squares and 32 white squares to be covered.

The Mutilated Checkerboard Problem



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“A Tough Nut for Proof Procedures” [McCarthy1964]

McCarthy 1964, Version a

1. $S(1,2) \wedge S(2,3) \wedge S(3,4) \wedge S(4,5)$
   $\wedge\ S(5,6) \wedge S(6,7) \wedge S(7,8)$
2. $S(x,y) \Rightarrow L(x,y)$
3. $L(x,y) \wedge L(y,z) \Rightarrow L(x,z) \wedge \neg S(x,z)$
4. $L(x,y) \Rightarrow \neg E(x,y)$
5. $E(x,x)$

} properties of $1,\ldots,8$, $S(x,y) \equiv (y = x+1)$, $L \equiv <$, $E \equiv =$

McCarthy 1964, Version a

$$\left.\begin{array}{rl}
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\end{array}\right\}$$

properties of $1,\ldots,8$,
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$$\left.\begin{array}{rl}
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7. & G^1(x,y) \Rightarrow \neg(G^2(x,y) \lor G^3(x,y) \lor G^4(x,y) \lor G^5(x,y)) \\
8. & G^2(x,y) \Rightarrow \neg(G^3(x,y) \lor G^4(x,y) \lor G^5(x,y)) \\
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$G^1,\ldots,G^4$: placement of dominoes,
G^5: uncovered

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} $G^1,\ldots,G^4$: placement of dominoes, G^5: uncovered

11. $G^5(1,1) \wedge G^5(8,8)$
12. $G^5(x,y) \Rightarrow (E(1,x) \wedge E(1,y)) \vee (E(8,x) \wedge E(8,y))$

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13. $S(x_1,x_2) \Rightarrow (G^1(x_1,y) \equiv G^3(x_2,y))$
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} adjacency of dominoes

15. $\neg G^3(1,y) \wedge \neg G^1(8,y) \wedge \neg G^2(x,8) \wedge \neg G^4(x,1)$ } border of the board

Paulson 1996

Definitions

$$
\begin{aligned}
less\_than(m) \quad &:= \quad \{i \in \mathbb{N} | i < m\} \\
board \quad &:= \quad less\_than(2 \cdot s(m)) \times less\_than(2 \cdot s(n)) \\
tiling(A) : set(set(\alpha)) \quad &:= \quad \{\} \in tiling(A) \wedge (a \in A \wedge t \in tiling(a) \\
&\qquad \wedge a \cap t = \{\}) \\
&\qquad \Rightarrow a \cup t \in tiling(a) \\
domino : set(set(\mathbb{N} \times \mathbb{N})) \quad &:= \quad \{(i, j), (i, s(j))\} \in domino \wedge \\
&\qquad \{(i, j), (s(i), j)\} \in domino
\end{aligned}
$$

Theorem

$$(board \setminus \{(0, 0)\}) \setminus \{(s(2 \cdot m), s(2 \cdot n))\} \notin tiling(domino)$$

Huet 1996

B, W sets

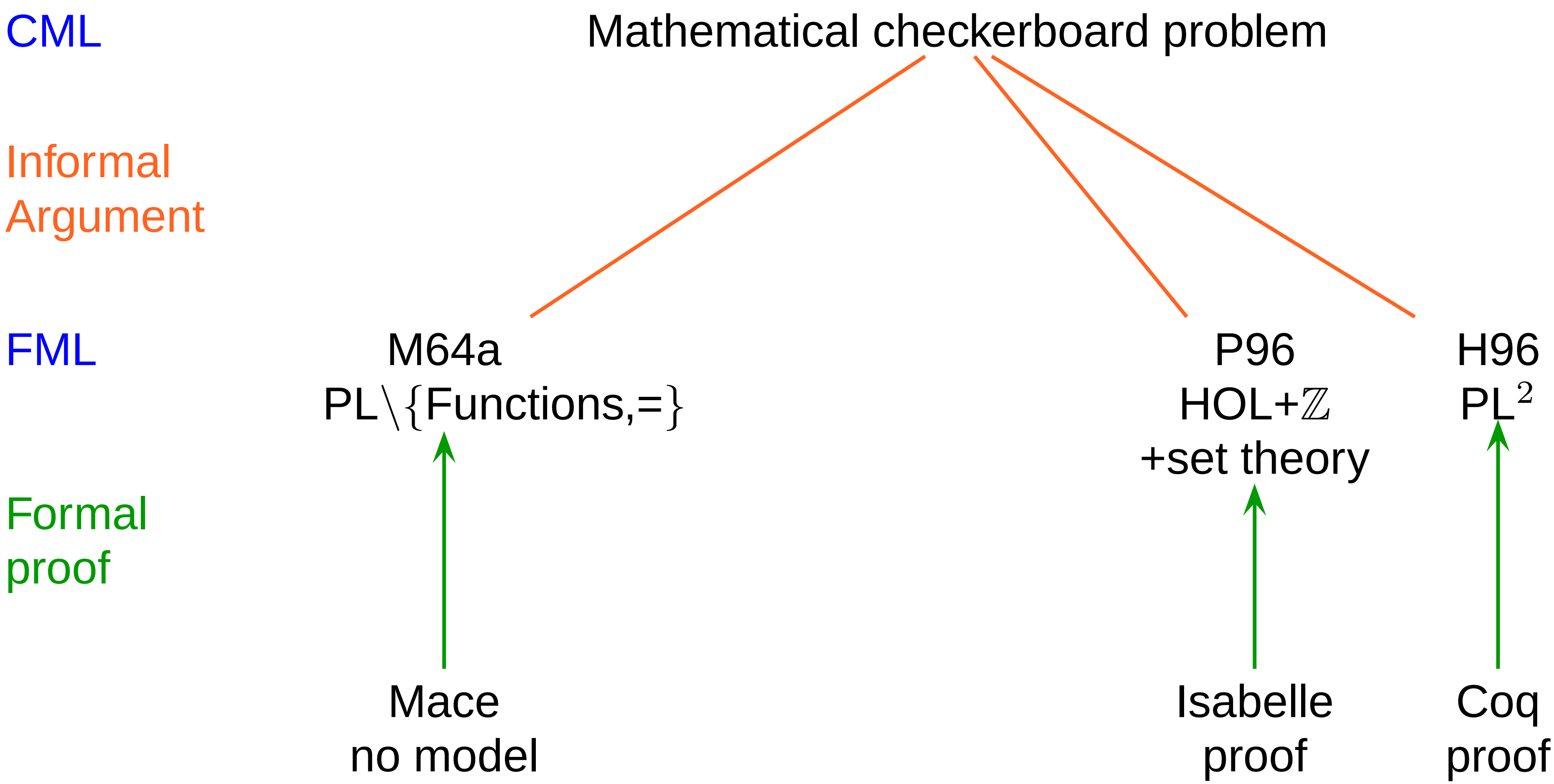
$Board : B \to W$

$Domino : W \to B$

Theorem

$injective(Board) \wedge injective(Domino) \wedge finite(B) \Rightarrow surjective(Domino)$

Overview of Formalisations



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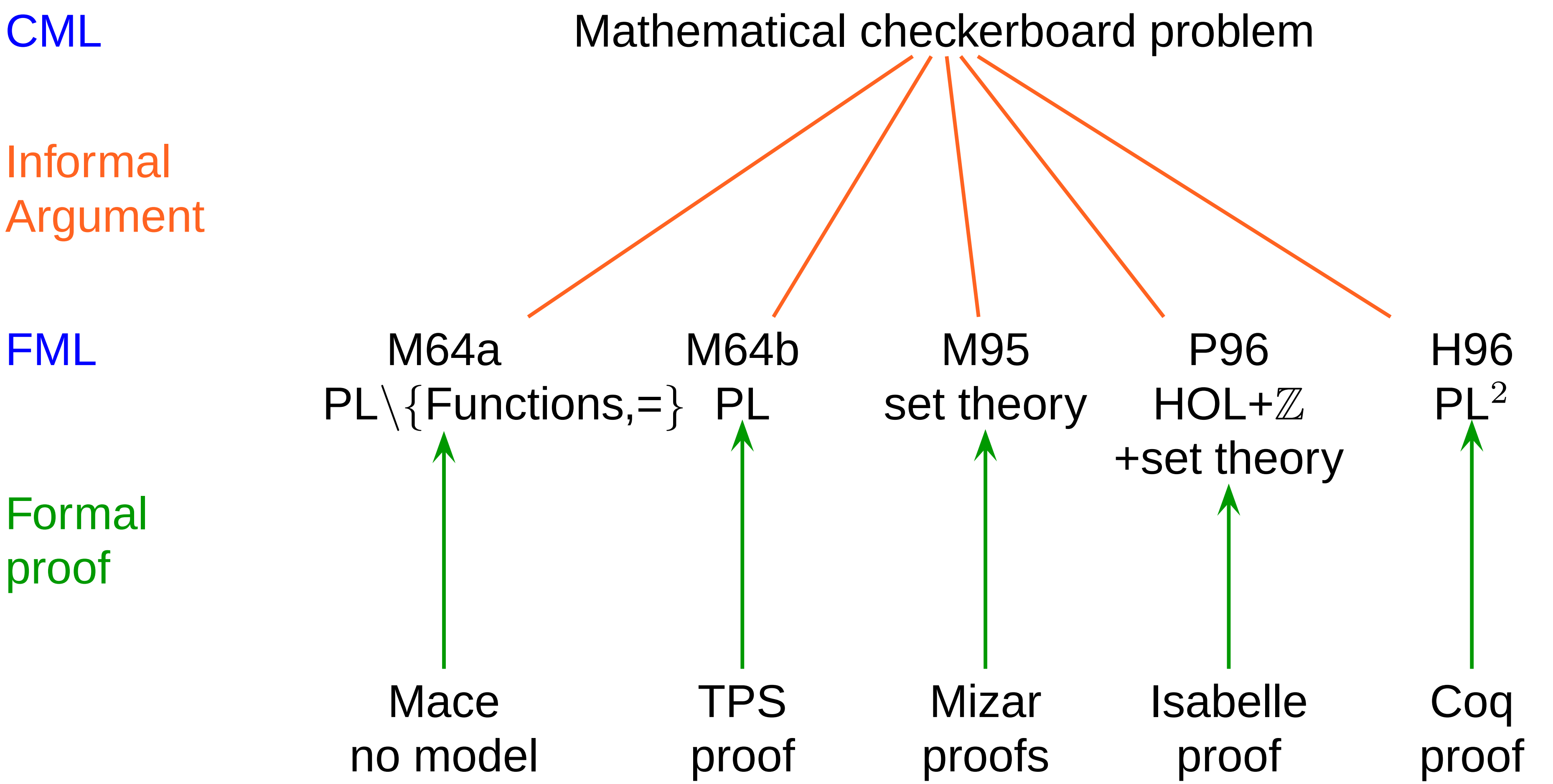

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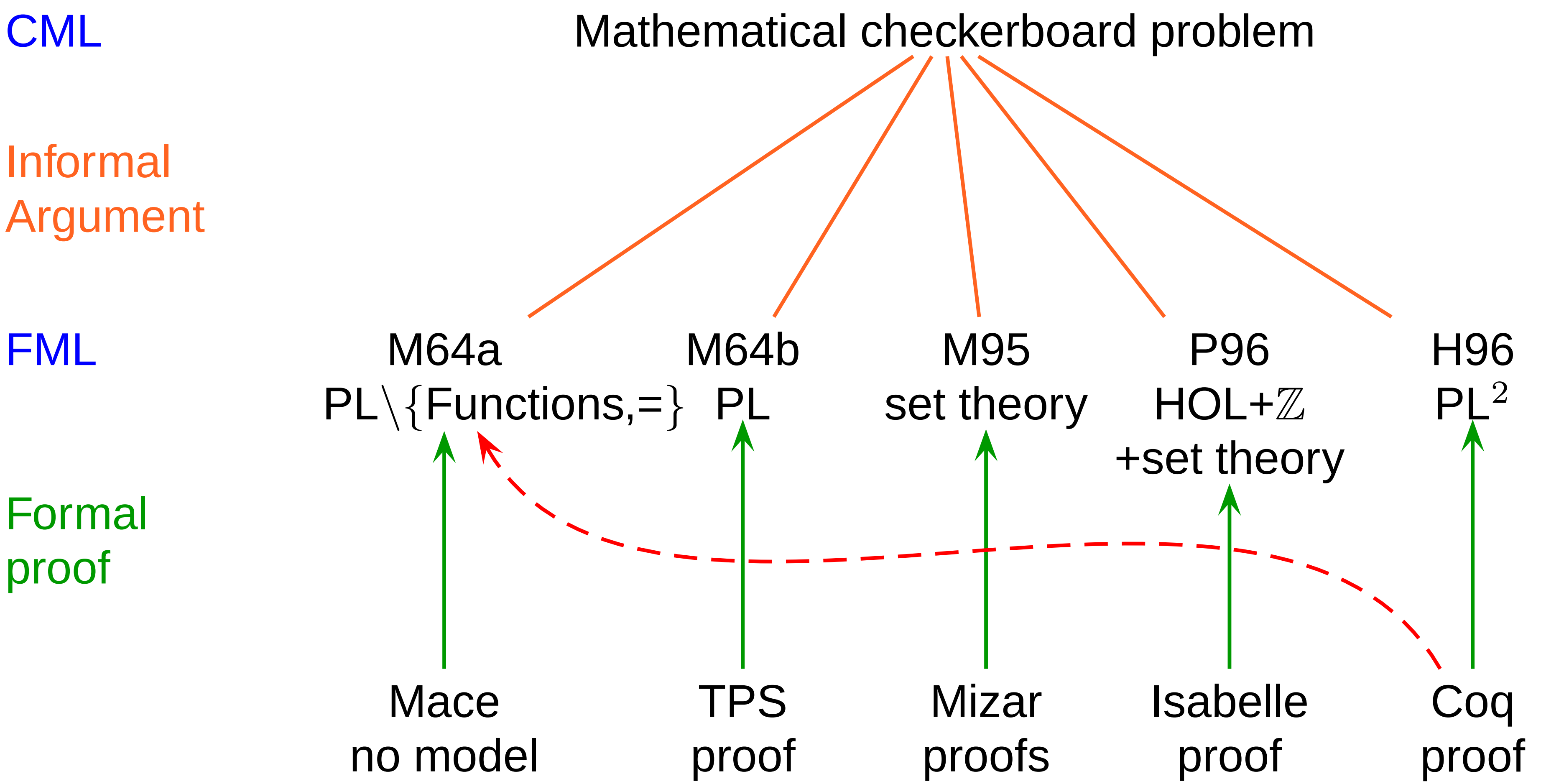

Signature M64a $\leftrightarrow$ H96

M64a	H96
$1, \ldots, 8$	B, W
$S(x, y)$	$Board : B \to W$
$E(x, y)$	$Domino : W \to B$
$L(x, y)$	injective
$G^1, \ldots, G^5$	surjective
	finite

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Formalisation of concepts?

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- Even more complex: which theorem to apply?

Formalisation of concepts depends on

- the formal language,
- automation provided by the system,
- theory hierarchy,
- existing formalisations (how to look for it?),
- purpose,
- personal style and preferences.