# Progress Report on Leo-II, an Automatic Theorem Prover for Higher-Order Logic


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LEO-II
An Effective Higher-Order Theorem Prover

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# Talk Outline

- Background
- LEO-II as Interactive Proof Assistant
- Automatic Proof Search
- Cooperation with other Reasoning Systems
- Term Sharing and Term Indexing
- First Experiments

# LEO-II Background

# Project Objectives

LEO-II
An Effective Higher-Order Theorem Prover

- Automatic theorem prover
  - resolution based HO reasoning
  - standalone system; implemented in Objective CAML
  - cooperation with specialist provers, e.g. FO ATPs
  - term sharing and term indexing
  - novel system architecture(s)
- Interactive proof assistant
- Cooperation with interactive proof assistants (not yet)
  - e.g. Isabelle/HOL
  - to support automatic proving of subproblems
  - for verification of own proof objects
- Problem representation language: TPTP THF Syntax

# LEO-II Basics

- Logic
  - classical higher-order logic (Church's simple type theory)
  - base types other than $\iota$ and $o$ can be specified
  - (strongly) limited support for polymorphism
- Syntax and Notation
  - typed variables: $X_\alpha, Y_\beta, Z_\gamma, X^1_\beta, X^2_\gamma \ldots$
  - typed constants: $c_\alpha, f_{\alpha\to\beta}, \ldots$
    including: $T_o$, $F_o$, $\neg_{o\to o}$, $\vee_{o\to o\to o}$, $\Pi_{(\alpha\to o)\to o}$, $=_{\alpha\to\alpha\to o}$
  - other logical connectives are defined as usual
  - abstraction and application terms defined as usual
- Target Semantics
  - Henkin models
  - $== (\lambda X.\lambda Y.Y = X)$

# Literals, Uni-Constraints, Clauses

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- Clauses and Literals

$$\mathcal{C}_1 : [\mathbf{A}_{\mathsf{o}}]^{=\mathsf{T}}, [\mathbf{B}_{\mathsf{o}}]^{=\mathsf{F}}, [\mathbf{C}_\alpha =^\alpha \mathbf{D}_\alpha]^{=\mathsf{T}}, [\mathbf{F}_\alpha =^\alpha \mathbf{G}_\alpha]^{=\mathsf{F}}$$

- Literal atoms are always kept in $\beta\eta$-normal form
- Negative equation literals: *unification constraints*.

$$\mathcal{C}_2 : [\mathbf{A}_{\mathsf{o}}]^{=\mathsf{T}}, [\mathbf{F}_\alpha =^\alpha \mathbf{G}_\alpha]^{=\mathsf{F}} \text{ corresponds to } (\mathbf{F}_\alpha =^\alpha \mathbf{G}_\alpha) \Rightarrow \mathbf{A}_{\mathsf{o}}$$

  - explains the name 'unification constraint'
  - $\mathbf{F}_\alpha$ and $\mathbf{G}_\alpha$ have a free variable at head position: *flex-flex*.
  - only one has a free variable at head position: *flex-rigid*.

# Literals, Uni-Constraints, Clauses

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- HO unification / pre-unification undecidable and infinitary
- HO pre-unification semi-decidable
- Example of inifinite number of pre-unifiers (H variable, f and a constants)

$$[\mathsf{H}_{\iota\to\iota}(\mathsf{f}_{\iota\to\iota}\mathsf{a}_\iota) = \mathsf{f}_{\iota\to\iota}(\mathsf{H}_{\iota\to\iota}\mathsf{a}_\iota)]^{=\mathsf{F}}$$

$$\mathsf{H} \longleftarrow \lambda \mathsf{x}.\,\underbrace{\mathsf{f}(\mathsf{f}\ldots(\mathsf{f}}_{\mathsf{n}\geq 0}\,\mathsf{x})\ldots)$$

- LEO-II operates with depth bounded pre-unification
- Definition of empty clause (modulo flex-flex pairs)

$$\mathcal{C} : [\mathsf{F}]^{=\mathsf{T}},\ \underbrace{[\mathbf{A}^1_\alpha =^\alpha \mathbf{B}^1_\alpha]^{=\mathsf{F}}, \ldots, [\mathbf{A}^{\mathsf{n}}_\alpha =^\alpha \mathbf{B}^{\mathsf{n}}_\alpha]^{=\mathsf{F}}}_{\text{only flex-flex unification constraints allowed}}$$

# LEO-II's Input Language: TPTP THF

- developed together with Geoff Sutcliffe, Florian Rabe, Allen van Gelder, Chad Brown, and others
- supports exchange of HO problems between systems
- THF core (THF0) covers at least simple type theory
- THF0 will be released soon

`http://www.cs.miami.edu/~tptp/TPTP/Proposals/THF.html`

# Example 1

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- Definitions

$$\text{reflexive} \stackrel{\text{def}}{=} \lambda R_{\iota\to\iota\to o}. \forall X_\iota. (R\ X\ X)$$

$$\text{symmetric} \stackrel{\text{def}}{=} \lambda R_{\iota\to\iota\to o}. \forall X_\iota. \forall Y_\iota. (R\ X\ Y) \Rightarrow (R\ Y\ X)$$

$$\text{transitive} \stackrel{\text{def}}{=} \lambda R_{\iota\to\iota\to o}. \forall X_\iota. \forall Y_\iota. \forall Z_\iota. ((R\ X\ Y) \land (R\ Y\ Z)) \Rightarrow (R\ X\ Z)$$

$$\text{equiv\_rel} \stackrel{\text{def}}{=} \lambda R_{\iota\to\iota\to o}. (\text{reflexive } R) \land (\text{symmetric } R) \land (\text{transitive } R)$$

- Theorem

$$\exists R_{\iota\to\iota\to o}. \neg(\text{equiv\_rel } R)$$

- Example solutions:

$$\{(x, y) | x \neq y\} \quad \text{represented by} \quad \lambda X_\iota. \lambda Y_\iota. \neg(X = Y)$$

$$\{(x, y) | \text{false}\} \quad \text{represented by} \quad \lambda X_\iota. \lambda Y_\iota. F$$

# THF Example 1

```
thf(reflexiv,definition,
    (reflexive :=
     (^[R:($i>($i>$o))]: (![X:$i]: ((R @ X) @ X))))).

thf(symmetric,definition,
    (symmetric :=
     (^[R:($i>($i>$o))]: (![X:$i,Y:$i]:
      ((R @ X) @ Y) => ((R @ Y) @ X))))).

thf(transitive,definition,
    (transitive :=
     (^[R:($i>($i>$o))]: (![X:$i,Y:$i,Z:$i]:
      ((((R @ X) @ Y) & ((R @ Y) @ Z)) => ((R @ X) @ Z)))))).

thf(equiv_rel,definition,
    (equiv_rel :=
     (^[R:($i>($i>$o))]:
      (reflexive @ R) & (symmetric @ R) & (transitive @ R)))).
thf(test,theorem,(?[R:($i>($i>$o))]: ~(equiv_rel @ R))).
```

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An Effective Higher-Order Theorem Prover

LEO-II as Interactive Proof Assistant

# LEO-II as Interactive Proof Assistant

- Interactive proof assistant for simple type theory
- Proof kernel: extensional-higher order resolution
- What is this good for?
  - teaching of higher-order reasoning, higher-order unification, and higher-order term data structures
  - debugging of calculus, strategies, heuristics, system architecture(s)
- However, main project goal is proof automation

```
LEO-II> help
* The list of interactive LEO-II commands is:
*  ***** interactive LEO-II calculus rules *****
*   bool <cl>                     - applies boolean extensionality to a clause
[...]
*   cnf-exhaustive <cl>           - exhaustive clause normalisation of a clause
[...]
*   res <cl1> <cl2>               - resolution between two clauses
[...]
*  ***** general commands *****
*   help                          - displays help screen;
*                                   type help <command> for help about <command>
*   analyze-index                 - displays information on the global index
[...]
*   clause-to-fotptp <cl>         - translates a clause to FOTPTP FOF syntax
[...]
*   flag-fo-translation           - determines the fo-translation to be used
*   flag-max-clause-count <max> - sets an upper limit for generating clauses
[...]
*   prove                         - starts automated proof search
*   prove-directory <dir>         - applies LEO-II to all files in a directory
*   prove-directory-with-fo-atp <dir> <prover> - applies LEO-II (with FO ATP)
*                                   to all files in a directory
*   prove-with-fo-atp             - starts automated proof search (with FO ATP)
*   read-problem-string <str>     - reads a problem string in THF syntax
*   read-problem-file <file>      - reads a problem in THF syntax from a file
[...]
*   quit                          - type this if you have enough of LEO-II
LEO-II>
```

Automatic Proof Search

# Problem Initialization

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- Given:
  definitions $\mathbf{D}_1, \ldots, \mathbf{D}_\mathsf{n}$, axioms $\mathbf{A}_1, \ldots, \mathbf{A}_\mathsf{n}$, conjecture $\mathbf{C}$
- Initialization leads to

$$\mathcal{C}_1 : [\mathbf{A}_1]^{=\mathsf{T}} \quad \ldots \quad \mathcal{C}_\mathsf{n} : [\mathbf{A}_\mathsf{n}]^{=\mathsf{T}} \qquad \mathcal{C}_{\mathsf{n}+1} : [\mathbf{C}]^{=\mathsf{F}}$$

- For our example problem we obtain

$$\mathcal{C}_1 : [\exists \mathsf{R}_{\iota\to\iota\to\mathsf{o}}. \neg(\mathsf{equiv\_rel}\ \mathsf{R})]^{=\mathsf{F}}$$

- What happens with the definitions $\mathbf{D}_1, \ldots, \mathbf{D}_\mathsf{n}$?
  - they are not explicitly represented as clauses
  - they are implicitly maintained as rewrite rules

# Example 1 (Contd.)

```
LEO-II> read-problem-file ../problems/SIMPLE-MATHS-5.thf
[...]
LEO-II> show-state
SIGNATURE:
 <base types>  $i  $o
 <type variables> 'A
 <fixed logical symbols>
  false (false) : $o
  [...]
 <defined symbols>
  and (&) : ^ [X:$o,Y:$o] : (~ ((~ X) | (~ Y)))
  [...]
  equiv_rel (equiv_rel) :
    ^ [R:$i>($i>$o)] :
        ((reflexive @ R) & ((symmetric @ R) & (transitive @ R)))
  [...]
 <uninterpreted symbols (upper case: free variables; lower case: constants)>
INDEX:   [...]
ACTIVE: [
2:[ 0:<? [R:$i>($i>$o)] : (~ (equiv_rel @ R)) = $false>-w(1)-i()]
  -mln(1)-w(1)-i(neg_input 1)-fv([ ])
]
PASSIVE: []
[...]
FLAGS:  [...]
LEO-II>
```

# Definition Unfolding

- currently LEO-II simultaneously unfolds all definitions before starting proof search
- thereby it benefits from the shared term data structures and the index
- delayed and stepwise definition unfolding, which is needed to successfully prove certain theorems, is future work

# Example 1 (Contd.)

```
LEO-II> unfold-defs-exhaustive
[
2:[ 0:<? [R:$i>($i>$o)] : (~ (equiv_rel @ R)) = $false>-w(1)-i() ]
  -mln(1)-w(1)-i(neg_input 1)-fv([ ])
]--- unfold-defs --->
 [
3:[ 0:<~ (! [x0:$i>($i>$o)] : (~ (~ (~ ((~ (! [x1:$i] :
       ((x0 @ x1) @ x1))) | (~ (~ ((~ (! [x1:$i,x2:$i] :
       ((~ ((x0 @ x1) @ x2)) | ((x0 @ x2) @ x1)))) |
       (~ (! [x1:$i,x2:$i,x3:$i] : ((~ (~ ((~ ((x0 @ x1) @ x2)) |
       (~ ((x0 @ x2) @ x3))))) | ((x0 @ x1) @ x3))))))))))))
       = $false>-w(1)-i() ]
  -mln(1)-w(1)-i(unfold_def 2)-fv([ ])
]
LEO-II>
```

# Clause Normalization

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- CNF rules provided for logical primitives: $\mathsf{T}, \mathsf{F}, \neg, \vee, \Pi^{\alpha}$ and $=^{\alpha}$
  - speciality (extensionality in CNF normalization)

$$\frac{\mathcal{C}, [\mathbf{F}_{\beta\to\gamma} = \mathbf{G}_{\beta\to\gamma}]^{=\alpha}}{\mathcal{C}, [\forall X_{\beta} . \mathbf{F}\, X = \mathbf{G}\, X]^{=\alpha}} =^{\mathsf{T},\mathsf{F}}_{\beta\to\gamma} \qquad \frac{\mathcal{C}, [\mathbf{F}_{o} = \mathbf{G}_{o}]^{=\alpha}}{\mathcal{C}, [\mathit{unfold}(\mathbf{F}_{o} \Leftrightarrow \mathbf{G}_{o})]^{=\alpha}} =^{\mathsf{T},\mathsf{F}}_{o}$$

  - otherwise CNF normalization still quite naive
- Normalization of clause 3 leads to (the $V^i$ are free variables)

$$\mathcal{C}_{15} : [V^0\, V^1\, V^1]^{=\mathsf{T}} \qquad \mathcal{C}_{25} : [V^0\, V^1\, V^2]^{=\mathsf{T}}, [V^0\, V^2\, V^1]^{=\mathsf{F}}$$

$$\mathcal{C}_{31} : [V^0\, V^1\, V^2]^{=\mathsf{F}}, [V^0\, V^2\, V^3]^{=\mathsf{F}}, [V^0\, V^1\, V^3]^{=\mathsf{T}}$$

# Example 1 (Contd.)

```
LEO-II> cnf-exhaustive 3
3:[ 0:<~ (! [x0:$i>($i>$o)] : (~ (~ (~ ((~ (! [x1:$i] :
       ((x0 @ x1) @ x1))) | (~ (~ ((~ (! [x1:$i,x2:$i] :
       ((~ ((x0 @ x1) @ x2)) | ((x0 @ x2) @ x1)))) |
       (~ (! [x1:$i,x2:$i,x3:$i] : ((~ (~ ((~ ((x0 @ x1) @ x2)) |
       (~ ((x0 @ x2) @ x3))))) | ((x0 @ x1) @ x3))))))))))))
       = $false>-w(1)-i() ]-mln(1)-w(1)-i(unfold_def 2)-fv([ ])
--- cnf-exhaustive --->
 [
13:[ 0:<(V_x0_1 @ V_x1_2) @ V_x1_2 = $true>-w(1)-i() ]
   -mln(1)-w(1)-i(cnf 11)-fv([ V_x1_2 V_x0_1 ])

25:[ 0:<(V_x0_1 @ V_x1_3) @ V_x2_5 = $false>-w(1)-i()
     1:<(V_x0_1 @ V_x2_5) @ V_x1_3 = $true>-w(1)-i() ]
   -mln(2)-w(2)-i(cnf 23)-fv([ V_x2_5 V_x1_3 V_x0_1 ])

31:[ 0:<(V_x0_1 @ V_x1_4) @ V_x2_6 = $false>-w(1)-i()
     1:<(V_x0_1 @ V_x1_4) @ V_x3_7 = $true>-w(1)-i()
     2:<(V_x0_1 @ V_x2_6) @ V_x3_7 = $false>-w(1)-i() ]
    -mln(3)-w(3)-i(cnf 30)-fv([ V_x3_7 V_x2_6 V_x1_4 V_x0_1 ])
]
LEO-II>
```

# Resolution & Co.

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- Resolution

$$\frac{\mathcal{C}, [\mathbf{A}]^{=\alpha} \quad \mathcal{D}, [\mathbf{B}]^{=\beta} \quad \alpha \neq \beta \in \{\mathsf{T}, \mathsf{F}\}}{\mathcal{C}, \mathcal{D}, [\mathbf{A} = \mathbf{B}]^{=\mathsf{F}}}\ \mathsf{res}$$

- Factorization

$$\frac{\mathcal{C}, [\mathbf{A}]^{=\alpha}, [\mathbf{B}]^{=\alpha}}{\mathcal{C}, [\mathbf{A}]^{=\alpha}, [\mathbf{A} = \mathbf{B}]^{=\mathsf{F}}}\ \mathsf{fac}$$

  - currently restricted to identical $\mathbf{A}$, $\mathbf{B}$ and handled via simplification rule

- Simplification
  - trivial factorization, deletion of tautologies, deletion of trivially unsatisfiable literals, etc.

# Extensional Pre-Unification

- Pre-unification

$$\frac{\mathcal{C}, [\mathbf{M}_{\alpha\to\beta} = \mathbf{N}_{\alpha\to\beta}]^{=F} \quad s_\alpha \text{ Sk. term}}{\mathcal{C}, [\mathbf{M}\ s = \mathbf{N}\ s]^{=F}} \text{ func}$$

$$\frac{\mathcal{C}, [(h_\alpha\ \overline{\mathbf{U}^n} = h_\alpha\ \overline{\mathbf{V}^n})]^{=F}}{\mathcal{C}, [\mathbf{U}^1 = \mathbf{V}^1]^{=F}, \ldots, [\mathbf{U}^n = \mathbf{V}^n]^{=F}} \text{ dec} \qquad \frac{\mathcal{C}, [\mathbf{A} = \mathbf{A}]^{=F}}{\mathcal{C}} \text{ triv}$$

$$\frac{\mathcal{C}, [(F_\gamma\ \overline{\mathbf{U}^n} = h\ \overline{\mathbf{V}^m})]^{=F} \quad \mathbf{G} \in \mathcal{AB}^h_\gamma}{\mathcal{C}, [F = \mathbf{G}]^{=F}, [F\ \overline{\mathbf{U}^n} = h\ \overline{\mathbf{V}^m}]^{=F}} \text{ flex-rigid}(F \leftarrow \mathbf{G})$$

$$\frac{\mathcal{C}, [X = \mathbf{A}]^{=F} \quad X \notin \mathbf{Free}(\mathbf{A})}{\{\mathbf{A}/X\}\mathcal{C}} \text{ subst}$$

- Extensional Pre-unification

$$\frac{\mathcal{C}, [\mathbf{M}_o = \mathbf{N}_o]^{=F}}{\mathcal{C}, [\mathit{unfold}(\mathbf{M}_o \Leftrightarrow \mathbf{N}_o)]^{=F}} \text{ bool}$$

  - clause normalization required after application of Bool

# Primitive Substitution

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- Primitive substitution (blind guessing of sets and relations)

$$\frac{\mathcal{C}, [\mathsf{P}\ \overline{\mathbf{U}^n}]^{=\alpha} \quad \mathbf{G} \in \mathcal{AB}^{\mathsf{T},\mathsf{F},\neg,\vee,\Pi^\alpha}}{\{\mathbf{G}/\mathsf{P}\}(\mathcal{C}, [\mathsf{P}\ \overline{\mathbf{U}^n}]^{=\alpha})}\ \text{prim-subst}(\mathsf{P} \leftarrow \mathbf{G})$$

- Example 1 (Contd.)

$$\mathcal{C}_{15} : [\mathsf{V}^0\ \mathsf{V}^1\ \mathsf{V}^1]^{=\mathsf{T}} \qquad \mathcal{C}_{25} : [\mathsf{V}^0\ \mathsf{V}^1\ \mathsf{V}^2]^{=\mathsf{T}}, [\mathsf{V}^0\ \mathsf{V}^2\ \mathsf{V}^1]^{=\mathsf{F}}$$

$$\mathcal{C}_{31} : [\mathsf{V}^0\ \mathsf{V}^1\ \mathsf{V}^2]^{=\mathsf{F}}, [\mathsf{V}^0\ \mathsf{V}^2\ \mathsf{V}^3]^{=\mathsf{F}}, [\mathsf{V}^0\ \mathsf{V}^1\ \mathsf{V}^3]^{=\mathsf{T}}$$

$$\frac{[\mathsf{V}^0\ \mathsf{V}^1\ \mathsf{V}^1]^{=\mathsf{T}} \quad \mathbf{G} \in \{\ldots, (\lambda \mathsf{Y}, \mathsf{Z}\text{.}\mathsf{F}), \ldots\}}{[\mathsf{F}]^{=\mathsf{T}}}\ \text{prim-subst}(\mathsf{V}^0 \leftarrow \lambda \mathsf{Y}, \mathsf{Z}\text{.}\mathsf{F})$$

# Paramodulation and Rewriting (ToDo)

- Literals as rewrite rules

$$\frac{[\mathbf{A}]^{=\alpha}, \mathcal{C} \quad \mathcal{D}[\mathbf{B}]_{\mathsf{pl}} \quad \sigma(\mathbf{A}) = \mathbf{B}}{\sigma(\mathcal{D}[\alpha]_{\mathsf{pl}}, \mathcal{C})} \text{ rewr-w-lit}$$

1. $[\mathbf{A}]^{=\alpha}$ is maximal in $[\mathbf{A}]^{=\alpha}, \mathcal{C}$ wrt. term ordering $>$
2. $\sigma(\mathcal{D}[\alpha]_{\mathsf{pl}}, \mathcal{C}) \not> \mathcal{D}[\mathbf{B}]_{\mathsf{pl}}$

- Paramodulation

$$\frac{[\mathbf{A} = \mathbf{C}]^{=\mathsf{T}}, \mathcal{C} \quad \mathcal{D}[\mathbf{B}]_{\mathsf{pl}} \quad \sigma(\mathbf{A}) = \mathbf{B}}{\sigma(\mathcal{D}[\mathbf{C}]_{\mathsf{pl}}, \mathcal{C})} \text{ para}$$

1. $[\mathbf{A} = \mathbf{C}]^{=\mathsf{T}}$ is max. in $[\mathbf{A} = \mathbf{C}]^{=\mathsf{T}}, \mathcal{C}$ wrt. term ordering $>$
2. $\mathbf{A} > \mathbf{C}$
3. $\sigma(\mathcal{D}[\alpha]_{\mathsf{pl}}, \mathcal{C}) \not> \mathcal{D}[\mathbf{B}]_{\mathsf{pl}}$

# Example 1 (Contd.)

```
LEO-II> read-problem-file ../problems/SIMPLE-MATHS-5.thf
[...]
LEO-II> prove
3 4 5 6 [...] 317 318
Eureka --- Thanks to Corina!
Here are the empty clauses
 [
319:[ 0:<$false = $true>-w(1)-i() ]-mln(1)-w(1)-i(sim 33)-fv([ ])
]
0.54003: Total Reasoning Time (../problems/SIMPLE-MATHS-5.thf)
```

```
LEO-II (Proof Found!)> show-derivation 319
**** Beginning of derivation protocol ****
1: (? [R:$i>($i>$o)] : (~ (equiv_rel @ R)))=$true
  ---  theorem(file(../problems/SIMPLE-MATHS-5.thf,[test]))
2: (? [R:$i>($i>$o)] : (~ (equiv_rel @ R)))
   =$false
  ---  neg_input 1
3: (~ (! [x0:$i>($i>$o)] : (~ (~ (~ ((~ (! [x1:$i] : ((x0 @ x1) @ x1))) |
   (~ (~ ((~ (! [x1:$i,x2:$i] : ((~ ((x0 @ x1) @ x2)) | ((x0 @ x2) @ x1)))) |
   (~ (! [x1:$i,x2:$i,x3:$i] : ((~ (~ ((~ ((x0 @ x1) @ x2)) |
   (~ ((x0 @ x2) @ x3))))) | ((x0 @ x1) @ x3))))))))))))
   =$false
  ---  unfold_def 2
4: [...]
  ---  cnf 4
6: [...]
  ---  cnf 5
7: [...]
  ---  cnf 6
8: [...]
  ---  cnf 7
10: (~ (! [x1:$i] : ((V_x0_1 @ x1) @ x1)))=$false
   ---  cnf 8
11: (! [x1:$i] : ((V_x0_1 @ x1) @ x1))=$true
   ---  cnf 10
13: ((V_x0_1 @ V_x1_2) @ V_x1_2)=$true  ---  cnf 11
33: ($false)=$true
   ---  prim-subst (V_x0_1 --> lambda [V21]: lambda [V22]: false) 13
319: ($false)=$true  ---  sim 33
**** End of derivation protocol ****
**** no. of clauses: 13 ****
LEO-II (Proof Found!)>
```

```
LEO-II (Proof Found!)> show-derivation-tstp  319
%------------------------------------------------------------------------------
% File       : ../problems/SIMPLE-MATHS-5.thf
[...]
% Comments   : *todo*
%------------------------------------------------------------------------------
%**** Beginning of derivation protocol in tstp ****

 thf(1,theorem,((? [R:$i>($i>$o)] : (~ (equiv_rel @ R)))=$true),
     file(../problems/SIMPLE-MATHS-5.thf,[test])).

 thf(2,plain,((? [R:$i>($i>$o)] : (~ (equiv_rel @ R)))=$false),
     inference(neg_input,[status(thm)],[1])).

[...]

 thf(33,plain,(($false)=$true),
     inference(prim-subst (V_x0_1-->lambda [V21]: lambda [V22]: false),
               [status(thm)],[13])).

 thf(319,plain,(($false)=$true),
     inference(sim,[status(thm)],[33])).

%**** End of derivation protocol in tstp ****
%**** no. of clauses in derivation: 13 ****
LEO-II (Proof Found!)>
```

# Proof Automation: ToDo List

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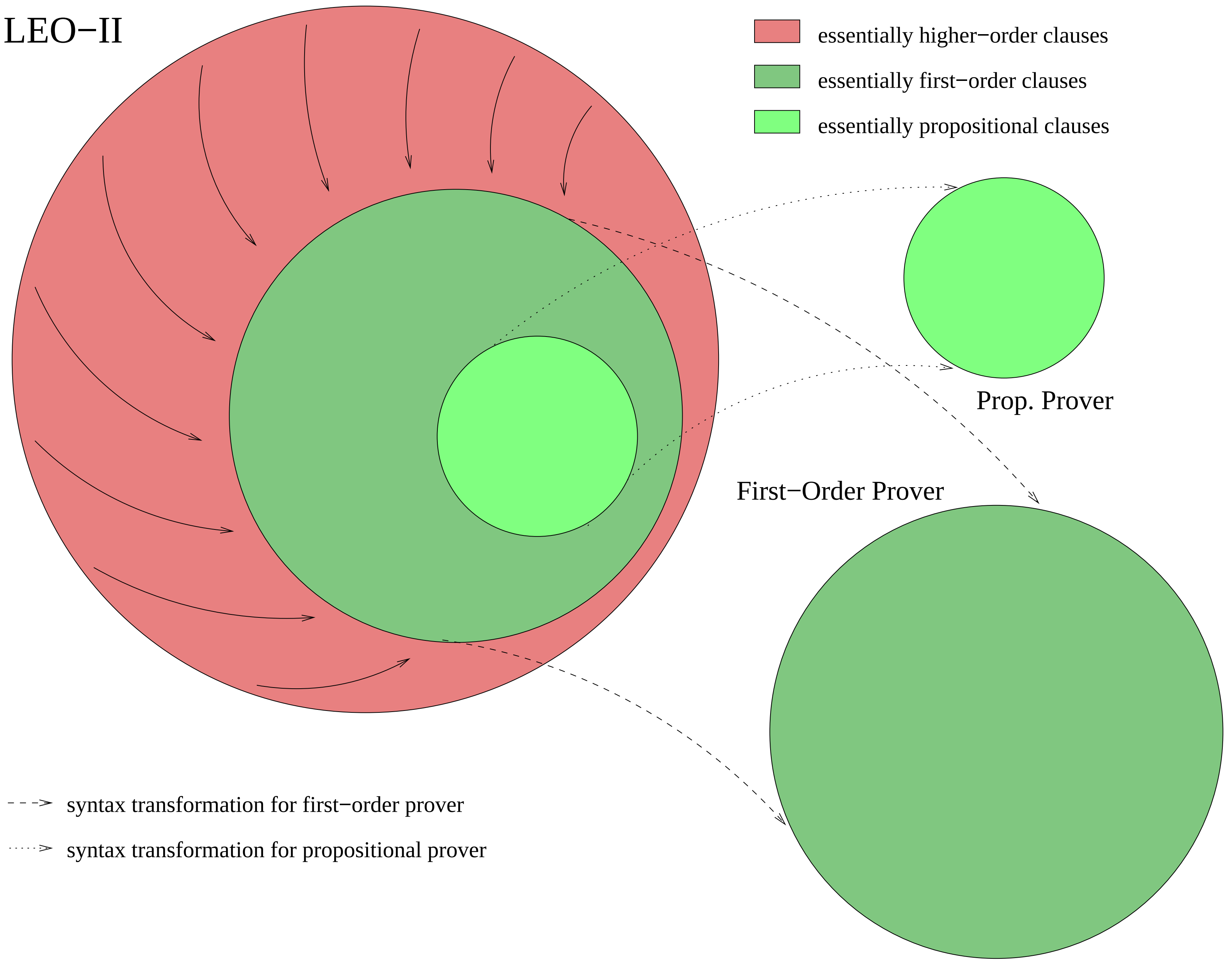
- term orderings
- efficient paramodulation and rewriting
- adapt overall calculus after adding them
- adapt heuristics and strategies
- efficient realization of remaining rules
- efficient subsumtion
- clever and efficient CNF normalization
- ...

Cooperation with FO-ATPs

# Cooperation with Other Provers





# Cooperation with Other Provers

LEO-II
An Effective Higher-Order Theorem Prover

- Provers supported (so far)
  - E, SPASS
- Translations supported so far
  - $@_\alpha$-FO-translation [Kerber94]:
    $(\mathsf{V}^0_{\iota\to\iota\to o}\ \mathsf{V}^1_\iota\ \mathsf{V}^1_\iota) \longrightarrow$
    $@_{(\iota\to o)\to\iota\to o}(@_{(\iota\to\iota\to o)\to\iota\to(\iota\to o)}(\mathsf{V}^0,\mathsf{V}^1),\mathsf{V}^1)$
  - fully typed FO-translation [Hurd02]:
    $(\mathsf{V}^0_{\iota\to\iota\to o}\ \mathsf{V}^1_\iota\ \mathsf{V}^1_\iota) \longrightarrow$
    $\mathsf{ti}(@(\mathsf{ti}(@(\mathsf{ti}(\mathsf{V}^0,\iota\to\iota\to o),\mathsf{ti}(\mathsf{V}^1,\iota)),\iota\to o),\mathsf{ti}(\mathsf{V}^1,\iota)),o)$

# Communication with FO-ATPs: TPTP FOF

```
[...]
 fof(leo_II_clause_54,axiom,(((~ lit(ti(at(ti(neg,ft(o,o)),
  ti(at(ti(at(ti(V88,ft(i,ft(i,o))),ti(V_x2_6,i)),ft(i,o)),
  ti(V_x3_7,i)),o)),o))) | (lit(ti(at(ti(neg,ft(o,o)),
  ti(at(ti(at(ti(V88,ft(i,ft(i,o))),ti(V_x1_4,i)),ft(i,o)),
  ti(V_x3_7,i)),o)),o)) | (~ lit(ti(at(ti(neg,ft(o,o)),
  ti(at(ti(at(ti(V88,ft(i,ft(i,o))),ti(V_x1_4,i)),ft(i,o)),
  ti(V_x2_6,i)),o)),o))))))).
[...]
```

# Cooperation with FO-ATPs: ToDo List

- add other provers, other systems
- use incremental provers
- provide more FO-translations
- parallel instead of sequential system architecture
- backtranslate proof objects
- . . .

# Perfect Term Sharing and Term Indexing

# Term Sharing and Term Indexing



- Term sharing and term indexing widely employed in FO ATPs
- Not much used in HO systems so far
- LEO-II
  - Perfectly shared term data structure
  - DeBruijn-notation for bound variables
  - Indexing of various structural properties
  - Index realized via hashtables
  - Many operations (not all yet!) supported by term indexing
  - We provide tools to analyze the datastructure and the index

# Perfect Term Sharing

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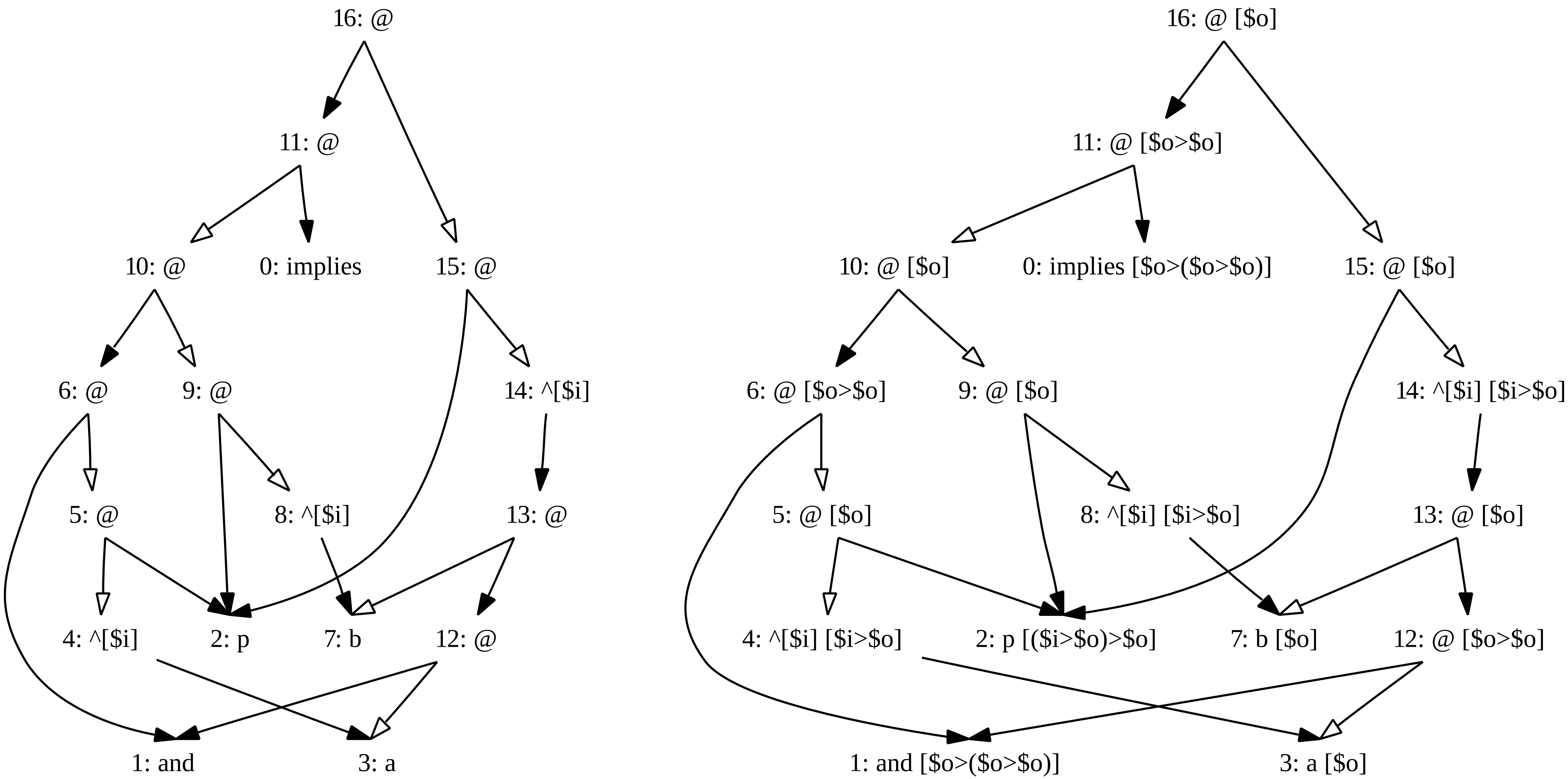
$$p_{\iota\to o}(\lambda X_\iota.a_o) \wedge p_{\iota\to o}(\lambda X_\iota.b_o) \Rightarrow p_{\iota\to o}(\lambda X_\iota.a_o \wedge b_o)$$

$$\Rightarrow [(\wedge\ (p_{\iota\to o}(\lambda X_\iota.a_o)))\ (p_{\iota\to o}(\lambda X_\iota.b_o))]\ [p_{\iota\to o}(\lambda X_\iota.a_o \wedge b_o)]$$



```
[...]
--- cnf-exhaustive --->
 [
13:[ 0:<(V_x0_1 @ V_x1_2) @ V_x1_2 = $true>-w(1)-i() ]
   -mln(1)-w(1)-i(cnf 11)-fv([ V_x1_2 V_x0_1 ])
25:[ 0:<(V_x0_1 @ V_x1_3) @ V_x2_5 = $false>-w(1)-i()
     1:<(V_x0_1 @ V_x2_5) @ V_x1_3 = $true>-w(1)-i() ]
   -mln(2)-w(2)-i(cnf 23)-fv([ V_x2_5 V_x1_3 V_x0_1 ])
31:[ 0:<(V_x0_1 @ V_x1_4) @ V_x2_6 = $false>-w(1)-i()
     1:<(V_x0_1 @ V_x1_4) @ V_x3_7 = $true>-w(1)-i()
     2:<(V_x0_1 @ V_x2_6) @ V_x3_7 = $false>-w(1)-i() ]
   -mln(3)-w(3)-i(cnf 30)-fv([ V_x3_7 V_x2_6 V_x1_4 V_x0_1 ])
]
[contd.]
```

```
[contd.]
LEO-II> inspect-symbol V_x0_1
Inspecting:
  node 315: V_x0_1
Type:
  $i>($i>$o)
Structure:
  symbol V_x0_1
Parents:
 - as function term:
  node 323: V_x0_1 @ V_x2_5
  node 326: V_x0_1 @ V_x1_4
  node 317: V_x0_1 @ V_x1_2
  node 331: V_x0_1 @ V_x2_6
  node 320: V_x0_1 @ V_x1_3
 total: 5 parents
[contd.]
```

```
[contd.]
Occurs in terms indexed with role:
  node 318: (V_x0_1 @ V_x1_2) @ V_x1_2
   (in Clause:13/0 max pos)
  node 322: (V_x0_1 @ V_x1_3) @ V_x2_5
   (in Clause:25/0 max neg)
  node 324: (V_x0_1 @ V_x2_5) @ V_x1_3
   (in Clause:25/1 max pos)
  node 328: (V_x0_1 @ V_x1_4) @ V_x2_6
   (in Clause:31/0 max neg)
  node 330: (V_x0_1 @ V_x1_4) @ V_x3_7
   (in Clause:31/1 max pos)
  node 332: (V_x0_1 @ V_x2_6) @ V_x3_7
   (in Clause:31/2 max neg)
 total: 6 terms
LEO-II>
```

```
LEO-II> read-problem-file ../problems/SIMPLE-MATHS-5.thf
[...]
LEO-II> analyze-index
------------- The Termset -------------
  0: symbol exists    : (’A>$o)>$o              1 parent(s)
  1: symbol neg       : $o>$o                   1 parent(s)
  2: symbol equiv_rel : ($i>($i>$o))>$o         1 parent(s)
  3: bound($i>($i>$o),0) : $i>($i>$o)           1 parent(s)
  4: appl(2,3)        : $o                      1 parent(s)
  5: appl(1,4)        : $o                      1 parent(s)
  6: abstr($i>($i>$o),5) : ($i>($i>$o))>$o      1 parent(s)
  7: appl(0,6)        : $o                  ---       [$i>($i>$o)/’A]
------------- End Termset -------------
------------- The Termset Analysis -------------
Heavily shared nodes:
Statistics:
 From 0 to 0 bindings: 1 node(s)
 From 0 to 1 bindings: 7 node(s)
Details of dense areas:
 From 0 to 0 bindings: 1 node(s)
 From 1 to 1 bindings: 7 node(s)
Sharing rate: 8 nodes with 7 bindings
Average sharing rate:                             0.875 bindings per node
Average term size:                                2.75
Average number of supernodes:                     2.25
Average number of supernodes (symbols):           2.66666666667
Average number of supernodes (nonprimitive terms): 1.5
Rate of term occurrences PST size / term size:     0.440298507463
Rate of symbol occurrences PST size / term size:   0.510204081633
Rate of bound occurrences PST size / term size:    0.636363636364
------------- End Termset Analysis -------------
LEO-II> prove
```

```
LEO-II> prove
3 4 [...] 317 318
Eureka --- Thanks to Corina!
Here are the empty clauses
 [319:[ 0:<$false = $true>-w(1)-i() ]-mln(1)-w(1)-i(sim 33)-fv([ ])]
LEO-II (Proof Found!)> analyze-index
------------- The Termset -------------
  0: symbol exists    : ('A>$o)>$o             6 parent(s)
  1: symbol neg       : $o>$o                  1 parent(s)
[...]
687: appl(684,686)    : $o                      ---
------------- End Termset -------------
------------- The Termset Analysis -------------
Heavily shared nodes:
 6 bindings: exists (node 0)
 48 bindings: neg (node 1)
 [...]
Statistics:
 [...]
 From 22 to 32 bindings: 6 node(s)
 From 39 to 48 bindings: 3 node(s)
Sharing rate: 688 nodes with 1042 bindings
Average sharing rate:                            1.51453488372 bindings per node
Average term size:                               8.49273255814
Average number of supernodes:                    6.44040697674
Average number of supernodes (symbols):          16.5897435897
Average number of supernodes (nonprimitive terms): 3.68412162162
Rate of term occurrences PST size / term size:     0.26666121598
Rate of symbol occurrences PST size / term size:   0.405684754522
Rate of bound occurrences PST size / term size:    0.506734951593
------------- End Termset Analysis -------------
LEO-II (Proof Found!)>
```

# Our Term Indexing Tools

- may be useful for other tasks
- need to be better exploited in LEO-II
- work out theoretical properties

First Experiments

# LEO+FO-ATPs vs. LEO-II+FO-ATPs

LEO-II
An Effective Higher-Order Theorem Prover

- Previous experiments published in:
  - C. Benzmüller, V. Sorge, M. Jamnik, and M. Kerber:
    **Combined Reasoning by Automated Cooperation.**
    Journal of Applied Logic, 2007. To appear.
  - C. Benzmüller, V. Sorge, M. Jamnik, and M. Kerber:
    **Can a Higher-Order and a First-Order Theorem Prover Cooperate?**
    Proc. of LPAR, LNAI 3452, pp. 415-431, 2005. Springer.
- TPTP SET Category:
  - Problems on Sets, Relations and Functions
  - Formulated in first-order set theory
  - Reformulated for experiments in simple type theory
- Computer used in old experiments: 2.4 GHz Xenon, 1GB memory
- In new experiments: 1.6 GHz Intel Pentium, 1 GB memory

# LEO+FO-ATPs vs. LEO-II+FO-ATPs



| | |
|---|---|
| SET171+3 | $\forall X_{o\alpha}, Y_{o\alpha}, Z_{o\alpha}. X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ |
| SET611+3 | $\forall X_{o\alpha}, Y_{o\alpha}. (X \cap Y = \emptyset) \Leftrightarrow (X \setminus Y = X)$ |
| SET624+3 | $\forall X_{o\alpha}, Y_{o\alpha}, Z_{o\alpha}. \mathtt{Meets}(X, Y \cap Z) \Leftrightarrow \mathtt{Meets}(X, Y) \vee \mathtt{Meets}(X, Z)$ |
| SET646+3 | $\forall x_{\alpha}, y_{\beta}. \mathtt{Subrel}(\mathtt{Pair}(x, y), (\lambda u_{\alpha}. T) \times (\lambda v_{\beta}. T))$ |
| SET670+3 | $\forall Z_{o\alpha}, R_{o\beta\alpha}, X_{o\alpha}, Y_{o\beta}. \mathtt{IsRelOn}(R, X, Y) \Rightarrow \mathtt{IsRelOn}(\mathtt{RestrictRDom}(R, Z), Z, Y)$ |

| | | |
|---|---|---|
| $\_ \in \_$ | := | $\lambda x_{\alpha}, A_{o\alpha}. [A x]$ |
| $\emptyset$ | := | $[\lambda x_{\alpha}. F]$ |
| $\_ \cap \_$ | := | $\lambda A_{o\alpha}, B_{o\alpha}. [\lambda x_{\alpha}. x \in A \wedge x \in B]$ |
| $\_ \cup \_$ | := | $\lambda A_{o\alpha}, B_{o\alpha}. [\lambda x_{\alpha}. x \in A \vee x \in B]$ |
| $\_ \setminus \_$ | := | $\lambda A_{o\alpha}, B_{o\alpha}. [\lambda x_{\alpha}. x \in A \vee x \notin B]$ |
| $\mathtt{Meets}(\_, \_)$ | := | $\lambda A_{o\alpha}, B_{o\alpha}. [\exists x_{\alpha}. x \in A \wedge x \in B]$ |
| $\mathtt{Pair}(\_, \_)$ | := | $\lambda x_{\alpha}, y_{\beta}. [\lambda u_{\alpha}, v_{\beta}. u = x \wedge v = y]$ |
| $\_ \times \_$ | := | $\lambda A_{o\alpha}, B_{o\beta}. [\lambda u_{\alpha}, v_{\beta}. u \in A \wedge v \in B]$ |
| $\mathtt{Subrel}(\_, \_)$ | := | $\lambda R_{o\beta\alpha}, Q_{o\beta\alpha}. [\forall x_{\alpha}, y_{\beta}. R x y \Rightarrow Q x y]$ |
| $\mathtt{IsRelOn}(\_, \_, \_)$ | := | $\lambda R_{o\beta\alpha}, A_{o\alpha}, B_{o\beta}. [\forall x_{\alpha}, y_{\beta}. R x y \Rightarrow x \in A \wedge y \in B]$ |
| $\mathtt{RestrictRDom}(\_, \_)$ | := | $\lambda R_{o\beta\alpha}, A_{o\alpha}. [\lambda x_{\alpha}, y_{\beta}. x \in A \wedge R x y]$ |

# LEO+FO-ATPs vs. LEO-II+FO-ATPs

| TPTP-Problem | Difficulty | Saturate | Mus cadet | E-Setheo | Vampire 7 | LEO | | | LEO-Bliksem | | | | | LEO-Vampire | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Strat. | Cl. | Time | Cl. | Time | FOcl | FOtm | GnCl | Cl. | Time | FOcl | FOtm | GnCl |
| SET014+4 | .67 | + | + | + | .01 | ST | 41 | .16 | 34 | 6.76 | 19 | .01 | 7 | 11 | 2.6 | .01 | .01 | 16 |
| SET017+1 | .56 | – | – | + | .03 | EXT | 3906 | 57.52 | 25 | 8.54 | 16 | .01 | 74 | 28 | 5.05 | 8 | .01 | 22 |
| SET066+1 | 1.00 | ? | – | – | – | – | – | – | 26 | 6.80 | 20 | .01 | 56 | 38 | 3.73 | 17 | .01 | 53 |
| SET067+1 | .56 | + | + | + | .04 | ST | 6 | .02 | 13 | .32 | 16 | .01 | 12 | 9 | .1 | 10 | .01 | 17 |
| SET076+1 | .67 | + | – | + | .00 | – | – | – | 10 | .47 | 18 | .01 | 35 | 12 | .97 | 12 | .01 | 27 |
| SET086+1 | .22 | + | – | + | .04 | ST | 2 | .01 | 2 | .01 | N/A | N/A | N/A | 2 | .01 | N/A | N/A | N/A |
| SET096+1 | .56 | + | – | + | .03 | – | – | – | 27 | 7.99 | 14 | .01 | 25 | 81 | 7.29 | 71 | 0.02 | 23 |
| SET143+3 | .67 | + | + | + | 68.71 | EIR | 37 | .38 | 33 | 7.93 | 18 | .01 | 19 | 8 | .31 | 9 | .01 | 9 |
| SET171+3 | .67 | + | + | – | 108.31 | EIR | 36 | .56 | 25 | 4.75 | 19 | .01 | 20 | 6 | .38 | 10 | .01 | 9 |
| SET580+3 | .44 | + | + | + | 14.71 | EIR | 25 | .19 | 6 | 2.73 | 8 | .01 | 13 | 8 | .23 | 12 | .01 | 4 |
| SET601+3 | .22 | + | + | + | 168.40 | EIR | 145 | 2.20 | 55 | 4.96 | 8 | .01 | 13 | 20 | 1.18 | 31 | .01 | 17 |
| SET606+3 | .78 | + | – | + | 62.02 | EIR | 21 | .33 | 17 | 10.8 | 15 | .01 | 5 | 5 | .27 | 5 | .01 | 3 |
| SET607+3 | .67 | + | + | + | 65.57 | EIR | 22 | .31 | 17 | 7.79 | 15 | .01 | 6 | 5 | .26 | 8 | .01 | 3 |
| SET609+3 | .89 | + | + | – | 161.78 | EIR | 37 | .60 | 26 | 6.50 | 19 | .01 | 17 | 6 | .49 | 10 | .01 | 9 |
| SET611+3 | .44 | + | – | + | 60.20 | EIR | 996 | 12.69 | 72 | 32.14 | 38 | .01 | 101 | 39 | 4.00 | 40 | 0.03 | 23 |
| SET612+3 | .89 | + | – | – | 113.33 | EIR | 41 | .54 | 18 | 3.95 | 6 | .01 | 7 | 8 | .46 | 11 | .01 | 9 |
| SET614+3 | .67 | + | + | – | 157.88 | EIR | 38 | .46 | 19 | 4.34 | 16 | .01 | 17 | 8 | .41 | 9 | .01 | 9 |
| SET615+3 | .67 | + | + | – | 109.01 | EIR | 38 | .57 | 17 | 3.59 | 6 | .01 | 9 | 6 | .47 | 8 | .01 | 9 |
| SET623+3 | 1.00 | ? | – | – | – | EXT | 43 | 8.84 | 23 | 9.54 | 10 | .01 | 14 | 9 | 2.27 | 10 | .01 | 8 |
| SET624+3 | .67 | + | – | + | .04 | ST | 4942 | 34.71 | 54 | 9.61 | 46 | .01 | 212 | 47 | 3.29 | 44 | .01 | 71 |
| SET630+3 | .44 | + | – | + | 60.39 | EIR | 11 | .07 | 6 | .08 | 8 | .01 | 4 | 4 | .05 | 6 | .01 | 10 |
| SET640+3 | .22 | + | – | + | 70.41 | EIR | 2 | .01 | 2 | .01 | N/A | N/A | N/A | 2 | .01 | N/A | N/A | N/A |
| SET646+3 | .56 | + | – | + | 59.63 | EIR | 2 | .01 | 2 | .01 | N/A | N/A | N/A | 2 | .01 | N/A | N/A | N/A |
| SET647+3 | .56 | + | – | + | 64.21 | EIR | 26 | .15 | 13 | .30 | 13 | .01 | 15 | 7 | .12 | 7 | .01 | 11 |
| SET648+3 | .56 | + | – | + | 64.22 | EIR | 26 | .15 | 14 | .30 | 13 | .01 | 16 | 7 | .12 | 9 | .01 | 3 |
| SET649+3 | .33 | – | – | + | 63.77 | EIR | 45 | .30 | 29 | 5.49 | 12 | .01 | 16 | 10 | .25 | 13 | .01 | 8 |
| SET651+3 | .44 | – | – | + | 63.88 | EIR | 20 | .10 | 11 | .16 | 10 | .01 | 11 | 7 | .09 | 8 | .01 | 2 |
| SET657+3 | .67 | + | – | + | 1.44 | EIR | 2 | .01 | 2 | .01 | N/A | N/A | N/A | 2 | .01 | N/A | N/A | N/A |
| SET669+3 | .22 | – | – | + | .34 | EI | 6 | .19 | 7 | .21 | N/A | N/A | N/A | 6 | .2 | N/A | N/A | N/A |
| SET670+3 | 1.00 | ? | – | – | – | EXT | 15 | .17 | 17 | .36 | 16 | .01 | 6 | 9 | .14 | 11 | .01 | 14 |
| SET671+3 | .78 | – | – | + | 218.02 | EIR | 78 | .64 | 7 | 2.71 | 10 | .01 | 14 | 13 | .47 | 11 | .01 | 9 |
| SET672+3 | 1.00 | ? | – | – | – | EXT | 27 | .4 | 30 | .70 | 21 | .01 | 11 | 10 | .23 | 12 | .01 | 14 |
| SET673+3 | .78 | – | – | + | 47.86 | EIR | 78 | .65 | 14 | 5.66 | 14 | .01 | 16 | 13 | .47 | 17 | .01 | 6 |
| SET680+3 | .33 | + | – | + | .07 | ST | 185 | .88 | 29 | 4.61 | 18 | .01 | 24 | 30 | 2.38 | 16 | .01 | 27 |
| SET683+3 | .22 | + | – | + | .06 | ST | 46 | .20 | 35 | 8.90 | 18 | .01 | 24 | 12 | .27 | 15 | .01 | 4 |
| SET684+3 | .78 | – | – | + | .33 | ST | 275 | 2.45 | 46 | 5.95 | 26 | .01 | 47 | 41 | 3.39 | 35 | .01 | 38 |
| SET686+3 | .56 | – | – | + | .11 | ST | 274 | 2.36 | 46 | 5.37 | 26 | .01 | 46 | 42 | 3.55 | 37 | .01 | 39 |
| SET716+4 | .89 | + | + | – | – | ST | 39 | .45 | 18 | 3.81 | 18 | .01 | 118 | 19 | .4 | 24 | 0.02 | 73 |
| SET724+4 | .89 | + | + | – | – | EXT | 154 | 2.75 | 18 | 7.21 | 15 | .01 | 23 | 10 | 1.91 | 14 | .01 | 20 |
| SET741+4 | 0.91 | ? | + | – | – | – | – | – | 21 | 92.76 | 22 | .01 | 104850 | 21 | 3.70 | 26 | .01 | 570 |
| SET747+4 | .89 | – | + | – | – | ST | 34 | .46 | 25 | 1.11 | 18 | .01 | 10 | 11 | 1.18 | 8 | .01 | 14 |
| SET752+4 | .89 | ? | + | – | – | – | – | – | 50 | 6.60 | 48 | .01 | 4363 | 50 | 516.0 | 48 | .01 | 4145104 |
| SET753+4 | .89 | ? | + | – | – | – | – | – | 15 | 3.07 | 12 | .01 | 19 | 12 | 1.64 | 12 | .01 | 47 |
| SET764+4 | .56 | + | + | + | .02 | EI | 2 | .01 | 2 | .01 | N/A | N/A | N/A | 2 | .01 | N/A | N/A | N/A |
| SET770+4 | .89 | + | + | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Average Total LEO-Vampire ($\surd$) = 12.963

# New Experiments with LEO-II

LEO-II
An Effective Higher-Order Theorem Prover

| | Fully Typed FO-Translation | | | $@_\alpha$-FO-Translation | | |
|---|---|---|---|---|---|---|
| Filename | Proof | LEO+E (s) | Total (s) | Proof | LEO+E (s) | Total (s) |
| SET014+4.thf | $\surd$ | 0.008 0.024 | 0.032 | $\surd$ | 0.008 0.013 | 0.021 |
| SET017+1.thf | $\surd$ | 0.040 0.035 | 0.075 | $\surd$ | 0.040 0.029 | 0.069 |
| SET066+1.thf | $\surd$ | 0.004 0.016 | 0.020 | $\surd$ | 0.008 0.018 | 0.026 |
| SET067+1.thf | $\surd$ | 0.008 0.036 | 0.044 | $\surd$ | 0.008 0.032 | 0.040 |
| SET076+1.thf | $\surd$ | 0.008 0.019 | 0.027 | $\surd$ | 0.004 0.013 | 0.017 |
| SET086+1.thf | $\surd$ | 0.004 | 0.004 | $\surd$ | 0.004 | 0.004 |
| SET096+1.thf | $\surd$ | 0.012 0.021 | 0.033 | $\surd$ | 0.004 0.016 | 0.020 |
| SET143+3.thf | $\surd$ | 0.028 0.037 | 0.065 | $\surd$ | 0.008 0.015 | 0.023 |
| SET171+3.thf | $\surd$ | 0.032 0.034 | 0.066 | $\surd$ | 0.008 0.019 | 0.027 |
| SET580+3.thf | $\surd$ | 0.240 0.083 | 0.323 | $\surd$ | 0.052 0.038 | 0.090 |
| SET601+3.thf | $\surd$ | 0.304 0.184 | 0.488 | $\surd$ | 0.044 0.028 | 0.072 |
| SET606+3.thf | $\surd$ | 0.024 0.034 | 0.058 | $\surd$ | 0.012 0.015 | 0.027 |
| SET607+3.thf | $\surd$ | 0.008 0.024 | 0.032 | $\surd$ | 0.008 0.015 | 0.023 |
| SET609+3.thf | $\surd$ | 0.044 0.047 | 0.091 | $\surd$ | 0.024 0.036 | 0.060 |
| SET611+3.thf | $\surd$ | 0.808 0.293 | 1.101 | $\surd$ | 0.084 0.026 | 0.110 |
| SET612+3.thf | $\surd$ | 0.040 0.041 | 0.081 | $\surd$ | 0.012 0.016 | 0.028 |
| SET614+3.thf | $\surd$ | 0.048 0.076 | 0.124 | $\surd$ | 0.016 0.034 | 0.050 |
| SET615+3.thf | $\surd$ | 0.044 0.056 | 0.100 | $\surd$ | 0.012 0.019 | 0.031 |
| SET623+3.thf | $\surd$ | 8.548 0.858 | 9.407 | $\surd$ | 1.008 0.064 | 1.072 |
| SET624+3.thf | $\surd$ | 0.048 0.092 | 0.140 | $\surd$ | 0.020 0.021 | 0.041 |
| SET630+3.thf | $\surd$ | 0.008 0.023 | 0.031 | $\surd$ | 0.008 0.018 | 0.026 |
| SET640+3.thf | $\surd$ | 0.012 | 0.012 | $\surd$ | 0.008 | 0.008 |
| SET646+3.thf | $\surd$ | 0.012 | 0.012 | $\surd$ | 0.020 | 0.020 |
| SET647+3.thf | $\surd$ | 0.016 0.020 | 0.036 | $\surd$ | 0.012 0.018 | 0.030 |
| ... | ... | ... | ... | ... | ... | ... |

# New Experiments with LEO-II

LEO-II
An Effective Higher-Order Theorem Prover

| Filename | Fully Typed FO-Translation Proof | LEO+E (s) | Total (s) | $@_\alpha$-FO-Translation Proof | LEO+E (s) | Total (s) |
|---|---|---|---|---|---|---|
| ... | ... | ... | ... | ... | ... | ... |
| SET648+3.thf | $\surd$ | 0.012 0.020 | 0.032 | $\surd$ | 0.016 0.015 | 0.031 |
| SET649+3.thf | $\surd$ | 0.016 0.024 | 0.040 | $\surd$ | 0.012 0.018 | 0.030 |
| SET651+3.thf | $\surd$ | 0.016 0.024 | 0.040 | $\surd$ | 0.012 0.018 | 0.030 |
| SET657+3.thf | $\surd$ | 0.012 | 0.012 | $\surd$ | 0.008 | 0.008 |
| SET669+3.thf | $\surd$ | 0.020 0.023 | 0.043 | $\surd$ | 0.020 0.019 | 0.039 |
| SET670+3.thf | $\surd$ | 0.028 0.039 | 0.067 | $\surd$ | 0.020 0.034 | 0.054 |
| SET671+3.thf | $\surd$ | 0.020 0.031 | 0.051 | $\surd$ | 0.016 0.019 | 0.035 |
| SET672+3.thf | $\surd$ | 0.016 0.020 | 0.036 | $\surd$ | 0.016 0.018 | 0.034 |
| SET673+3.thf | $\surd$ | 0.020 0.031 | 0.051 | $\surd$ | 0.020 0.019 | 0.039 |
| SET680+3.thf | $\surd$ | 0.020 0.032 | 0.052 | $\surd$ | 0.020 0.016 | 0.036 |
| SET683+3.thf | $\surd$ | 0.012 0.023 | 0.035 | $\surd$ | 0.032 0.034 | 0.066 |
| SET684+3.thf | $\surd$ | 0.028 0.041 | 0.069 | $\surd$ | 0.016 0.020 | 0.036 |
| SET716+4.thf | $\surd$ | 0.012 0.020 | 0.032 | $\surd$ | 0.008 0.019 | 0.027 |
| SET724+4.thf | $\surd$ | 0.012 0.022 | 0.034 | $\surd$ | 0.012 0.018 | 0.030 |
| SET741+4.thf | $\surd$ | 0.016 0.037 | 0.053 | $\surd$ | 0.012 0.017 | 0.029 |
| SET747+4.thf | $\surd$ | 0.012 0.024 | 0.036 | $\surd$ | 0.008 0.019 | 0.027 |
| SET752+4.thf | $\surd$ | 0.028 0.267 | 0.295 | $\surd$ | 0.020 0.056 | 0.076 |
| SET753+4.thf | $\surd$ | 0.016 0.021 | 0.037 | $\surd$ | 0.016 0.018 | 0.034 |
| SET764+4.thf | $\surd$ | 0.008 | 0.008 | $\surd$ | 0.008 | 0.008 |
| SET770+4.thf | — | | | — | | |
| | Average Total ($\surd$) = 0.312 | | | Average Total ($\surd$) = 0.062 | | |

# SET770+4

$$\forall R_{\alpha\to\alpha\to o}, Q_{\alpha\to\alpha\to o}.((\mathsf{equiv\_rel}\ R) \wedge (\mathsf{equiv\_rel}\ Q)) \Rightarrow$$

$$((\mathsf{equiv\_classes}\ R) = (\mathsf{equiv\_classes}\ Q) \vee (\mathsf{disjoint}\ (\mathsf{equiv\_classes}\ R)\ (\mathsf{equiv\_classes}\ Q)))$$

# Further Work

| Filename | Fully Typed FO-Translation | | | $@_\alpha$-FO-Translation | | |
| --- | --- | --- | --- | --- | --- | --- |
| | Proof | LEO+E (s) | Total (s) | Proof | LEO+E (s) | Total (s) |
| n-bit-adder-base.thf | $\surd$ | 0.399 12.240 | 12.640 | — | | |
| n-bit-adder-step.thf | — | | | — | | |

# Summary

- LEO-II: so far 12570 lines of OCAML code, easy to install
  - shared term datastructure, term indexing, inspection tools
  - TPTP THF/FOF parser
  - command line interface
  - calculus
  - proof objects, proof output
  - automated proof search
  - support tools for experiments
  - . . .
- Long list of future work
- Now we are entering the fascinating phase
- Biggest problem: stay focused

# Why no (full) Polymorphism?

LEO-II
An Effective Higher-Order Theorem Prover

- adds another dimension of complexity and non-determinism:

| $\wedge_{o\to o\to o}$ | $T_o$ | $F_o$ |
|---|---|---|
| $T_o$ | $T_o$ | $F_o$ |
| $F_o$ | $F_o$ | $F_o$ |

| $\lambda F_{o\to o}\lambda G_{o\to o}\lambda X_o.(G\ (F\ X))$ | $\lambda X_o.X_o$ | $\lambda X_o.T$ |
|---|---|---|
| $\lambda X_o.X_o$ | $\lambda X_o.X_o$ | $\lambda X_o.T$ |
| $\lambda X_o.T$ | $\lambda X_o.T$ | $\lambda X_o.T$ |

- general:

| $Op_{\alpha\to\alpha\to\alpha}$ | $A_\alpha$ | $B_\alpha$ |
|---|---|---|
| $A_\alpha$ | $A_\alpha$ | $B_\alpha$ |
| $B_\alpha$ | $B_\alpha$ | $B_\alpha$ |

$$\begin{aligned}\exists\alpha.\ &\exists Op_{\alpha\to\alpha\to\alpha}.\exists A_\alpha.\exists B_\alpha.\\ &A \neq B\\ &\wedge(OpAA) = A \wedge (OpAB) = B\\ &\wedge(OpBA) = B \wedge (OpBB) = B\end{aligned}$$

- negation and clause normalization ($A, B, Op$ are free variables):

$$\mathcal{E}_1 : [A_\alpha = B_\alpha]^{=T}, [(Op_{\alpha\to\alpha\to\alpha}A_\alpha A_\alpha) = A_\alpha]^{=F}, [(Op_{\alpha\to\alpha\to\alpha}A_\alpha B_\alpha) = B_\alpha]^{=F},$$
$$[(Op_{\alpha\to\alpha\to\alpha}B_\alpha A_\alpha) = B_\alpha]^{=F}, [(Op_{\alpha\to\alpha\to\alpha}B_\alpha A_\alpha) = B_\alpha]^{=F}$$

- blind guessing of instances for type variable $\alpha$ in combination with blind guessing of instances for term variable $Op$ required