

Exercise sheet 1

Semantics of Higher-Order Logics

(2007)

For exercises 1-3, let \mathcal{D} be the standard frame with $\mathcal{D}_o = \{\perp, \top\}$ and $\mathcal{D}_l = \{1\}$.

Exercise 1 Assume $(\mathcal{E}_\alpha)_{\alpha \in \mathcal{T}}$ is a standard frame with

$$\mathcal{E}_o = \{\perp, \top\}$$

$$\mathcal{E}_l = \{1\}$$

Prove: $\forall \alpha \in \mathcal{T} : \mathcal{E}_\alpha = \mathcal{D}_\alpha$

Solution: We argue by induction on types. At both base types we know

$$\mathcal{E}_l = \{1\} = \mathcal{D}_l$$

and

$$\mathcal{E}_o = \{\perp, \top\} = \mathcal{D}_o$$

For the induction step, we assume $\mathcal{E}_\alpha = \mathcal{D}_\alpha$ and $\mathcal{E}_\beta = \mathcal{D}_\beta$.

Since both \mathcal{D} and \mathcal{E} are standard frames,

$$\mathcal{D}_{\alpha\beta} = (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} = (\mathcal{E}_\alpha)^{\mathcal{E}_\beta} = \mathcal{E}_{\alpha\beta}.$$

That was easy!

Exercise 2 Prove: $\forall \alpha \in \mathcal{T} : \mathcal{D}_\alpha$ is finite.

Solution: We argue by induction on types. Obviously $\mathcal{D}_o = \{\perp, \top\}$ and $\mathcal{D}_l = \{1\}$ are both finite. For the induction step, we assume \mathcal{D}_α and \mathcal{D}_β are both finite. Since \mathcal{D} is a standard frame, we know $\mathcal{D}_{\alpha\beta} = (\mathcal{D}_\alpha)^{\mathcal{D}_\beta}$. Hence we calculate

$$|\mathcal{D}_{\alpha\beta}| = |\mathcal{D}_\alpha|^{|\mathcal{D}_\beta|} < \infty$$

Thus all the domains are finite.

Exercise 3 Define inductively an infinite set $\mathcal{T}^1 \subseteq \mathcal{T}$ s.t.

$$\forall \alpha \in \mathcal{T}^1 \quad |\mathcal{D}_\alpha| = 1$$

Solution: Let \mathcal{T}^1 be the least set of types such that

- $l \in \mathcal{T}^1$
- $(\alpha\beta) \in \mathcal{T}^1$ whenever $\alpha \in \mathcal{T}^1$ and $\beta \in \mathcal{T}$

Intuitively, \mathcal{T}^1 is the set of types of the form $(l\beta^n \cdots \beta^1)$ for $n \geq 0$ and arbitrary types β^1, \dots, β^n .

We can inductively prove $|\mathcal{D}_\alpha| = 1$ for each $\alpha \in \mathcal{T}^1$.

- Base case: $|\mathcal{D}_l| = 1$.
- Induction case: Assume the type is $(\alpha\beta)$ where $\alpha \in \mathcal{T}^1$ and $\beta \in \mathcal{T}$. Assume $|\mathcal{D}_\alpha| = 1$. (We do not assume $|\mathcal{D}_\beta| = 1$ – why not?) We calculate:

$$|\mathcal{D}_{\alpha\beta}| = |\mathcal{D}_\alpha|^{|\mathcal{D}_\beta|} = 1^{|\mathcal{D}_\beta|} = 1.$$

Exercise 4 Prove every functional Σ -evaluation is ξ -functional.

Solution: To show functionality implies ξ -functionality, let $\mathbf{M}, \mathbf{N} \in \text{wff}_\beta(\Sigma)$, an assignment φ and a variable X_α be given. Suppose $\mathcal{E}_{\varphi, [a/X]}(\mathbf{M}) = \mathcal{E}_{\varphi, [a/X]}(\mathbf{N})$ for every $a \in \mathcal{D}_\alpha$. We need to show $\mathcal{E}_\varphi(\lambda X. \mathbf{M}) = \mathcal{E}_\varphi(\lambda X. \mathbf{N})$. This follows from functionality since

$$\begin{aligned} \mathcal{E}_\varphi(\lambda X. \mathbf{M})@a &= \mathcal{E}_{\varphi, [a/X]}((\lambda X. \mathbf{M})X) = \mathcal{E}_{\varphi, [a/X]}(\mathbf{M}) \\ &= \mathcal{E}_{\varphi, [a/X]}(\mathbf{N}) = \mathcal{E}_{\varphi, [a/X]}((\lambda X. \mathbf{N})X) = \mathcal{E}_\varphi(\lambda X. \mathbf{N})@a \end{aligned}$$

for every $a \in \mathcal{D}_\alpha$.

Exercise 5 Let $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ be a functional Σ -evaluation, φ be an assignment into \mathcal{J} , $\mathbf{F} \in \text{wff}_{\alpha \rightarrow \beta}(\Sigma)$ and $X_\alpha \notin \text{Free}(\mathbf{F})$. Prove

$$\mathcal{E}_\varphi(\lambda X_\alpha. \mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F}).$$

Solution: Let $a \in \mathcal{D}_\alpha$ be given. Since $X_\alpha \notin \text{Free}(\mathbf{F})$, we have $\mathcal{E}_{\varphi, [a/X]}(\mathbf{F}) = \mathcal{E}_\varphi(\mathbf{F})$. Since \mathcal{E} respects β -equality, we can compute

$$\mathcal{E}_\varphi(\lambda X. \mathbf{F}X)@a = \mathcal{E}_{\varphi, [a/X]}((\lambda X. \mathbf{F}X)X) = \mathcal{E}_{\varphi, [a/X]}(\mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F})@a.$$

Generalizing over a , we conclude $\mathcal{E}_\varphi(\lambda X. \mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F})$ by functionality.

Exercise 6 Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$ be a Σ -model. Prove if either $\top, \perp \in \Sigma$ or $\neg \in \Sigma$, then v is surjective.

Solution: Suppose $\top, \perp \in \Sigma$. By $\mathfrak{L}_\top(\mathcal{E}(\top))$ and $\mathfrak{L}_\perp(\mathcal{E}(\perp))$, we have $v(\mathcal{E}(\top)) = \mathbf{T}$ and $v(\mathcal{E}(\perp)) = \mathbf{F}$. Thus v is surjective.

Suppose $\neg \in \Sigma$. Choose any $a \in \mathcal{D}_o$. We know $v(\mathcal{E}(\neg)@a) \neq v(a)$ by $\mathfrak{L}_\neg(\mathcal{E}(\neg))$. Thus v is surjective.

Exercise 7 Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$ be a Σ -model. Suppose either $\top, \perp \in \Sigma$ or $\neg \in \Sigma$. Prove \mathcal{M} satisfies \mathfrak{b} iff \mathcal{D}_o has two elements.

Solution: By the previous exercise, we know v is surjective. Thus \mathcal{M} satisfies property \mathfrak{b} iff v is bijective iff \mathcal{D}_o has two elements.

Exercise 8 Assume that the signature contains only the logical connective \supset and the quantifier Π^o . Construct a Σ -model \mathcal{M} such that

1. $\mathcal{M} \models \forall P_o. P$

Solution: There is a model such that $\mathcal{M} \models \forall P_o. P$. Let \mathcal{D} be the standard frame with $\mathcal{D}_o = \{\mathbf{T}\}$ and $\mathcal{D}_i = \{1\}$. Note that every \mathcal{D}_α has only one element. Let $@$ be the application operator. For every assignment φ and $\mathbf{A} \in \text{wff}_\alpha$, let $\mathcal{E}_\varphi(\mathbf{A})$ be the unique member of \mathcal{D}_α . It is easy to check \mathcal{E} is an evaluation function. Let $v : \mathcal{D}_o \rightarrow \{\mathbf{T}, \mathbf{F}\}$ be the inclusion function given by $v(\mathbf{T}) := \mathbf{T}$. It is easy to check $\mathfrak{L}_\supset((\mathcal{E}(\supset)))$ and $\mathfrak{L}_{\Pi^o}((\mathcal{E}(\Pi^o)))$ hold where $\mathcal{E}(\supset)$ is the unique element of \mathcal{D}_{ooo} (the ‘‘constantly constant true function’’) and $\mathcal{E}(\Pi^o)$ is the unique element of $\mathcal{D}_{o(oo)}$ (the ‘‘constant true function’’). Hence $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, v)$ is a Σ -model.

Now to check $\mathcal{M} \models \forall P_o. P$, simply note that $\mathcal{M} \models_{\varphi, [\mathbf{T}/P]} P$ and so $\mathcal{M} \models_{\varphi, [a/P]} P$ for all $a \in \mathcal{D}_o$.

Exercise 9 What are the weakest calculi \mathfrak{R}_* in which the following sentences can be derived? Please give the derivations.

1. $\forall X_o. \forall Y_o. X \vee Y \Leftrightarrow Y \vee X$
2. $\forall X_o. \forall Y_o. X \vee Y \doteq Y \vee X$
3. $\lambda X_o. \lambda Y_o. X \vee Y \doteq \lambda X_o. \lambda Y_o. Y \vee X$
4. $\vee \doteq \lambda X_o. \lambda Y_o. Y \vee X$