A Structured Set of Higher-Order Problems

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Research Topics of the ΩMEGA Group

Integrated mathematics assistance systems

. . . for formal methods, mathematics, and e-learning

- Mediation between “mathural” and "maschine mathematics”

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Talk: Mechanization of Church’s simple type theory; test problems

Test Problems for Theorem Provers

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- Are we proposing challenging HOL benchmark problems?

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- Are we proposing challenging HOL benchmark problems?
  - No!!!

Our Examples

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- (Some more challenging examples are also added)

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- Conclusion

HOL: Simple Types

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o	(truth values)
ι	(individuals)
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$(\alpha\beta\gamma)$ abbreviates $((\alpha\beta)\gamma)$

HOL: Simply Typed λ-Terms

Terms:

X_α	Variables ($\mathcal{V}$)
a_α	Parameters ($\mathcal{P}$)
c_α	Logical constants ($\mathcal{S}$)
$[F_{\alpha\beta}\, B_\beta]_\alpha$	Application
$[\lambda Y_\beta\, A_\alpha]_{\alpha\beta}$	λ-abstraction

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α-conversion	Changing Bound Variables	
β-reduction	$[[\lambda Y_\beta\, A_\alpha]\, B] \xrightarrow{\beta} [B/Y]A$	
η-reduction	$[\lambda Y_\beta\, [F_{\alpha\beta}\, Y]] \xrightarrow{\eta} F$	$(Y_\beta \notin \textbf{Free}(F))$

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Equality of terms: $\alpha\beta\eta$

Every term has a unique $\beta\eta$-normal form (up to α-conversion).

HOL: Logical Constants

Some logical constants which may be in $\mathcal{S}$:

- $\top_o$ – true
- $\bot_o$ – false
- $\neg_{oo}$ – negation
- $\vee_{ooo}$ – disjunction
- $\wedge_{ooo}$ – conjunction
- $\Rightarrow_{ooo}$ – implication
- $\Leftrightarrow_{ooo}$ – equivalence

HOL: Logical Constants

Some logical constants which may be in $\mathcal{S}$:

- $=^{\alpha}_{\mathsf{o}\alpha\alpha}$ – equality at type α
- $\Pi^{\alpha}_{\mathsf{o}(\mathsf{o}\alpha)}$ – universal quantification over type α
- $\Sigma^{\alpha}_{\mathsf{o}(\mathsf{o}\alpha)}$ – existential quantification over type α

Intuition:

- $[\Sigma^{\alpha}\,[\lambda X_{\alpha}\, C_{\mathsf{o}}]]$ is true iff $\{X_{\alpha} \mid C\}$ is nonempty.

HOL: Logical Constants

- $[A_o \lor B_o]$ means $[\lor_{ooo} A_o B_o]$
- $[A_o \land B_o]$ means $[\land_{ooo} A_o B_o]$
- $[A_o \Rightarrow B_o]$ means $[\Rightarrow_{ooo} A_o B_o]$
- $[A_o \Leftrightarrow B_o]$ means $[\Leftrightarrow_{ooo} A_o B_o]$
- $[A_\alpha =^\alpha B_\alpha]$ means $[=^\alpha_{o\alpha\alpha} A_\alpha B_\alpha]$
- $[\forall X_\alpha A_o]$ means $[\Pi^\alpha_{o(o\alpha)} \blacksquare \lambda X_\alpha A_o]$
- $[\exists X_\alpha A_o]$ means $[\Sigma^\alpha_{o(o\alpha)} \blacksquare \lambda X_\alpha A_o]$

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- $\ddot{=}$ . . . other definition of equality (e.g., see [Andrews02])

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We use $\stackrel{*}{=}$ in the following to refer to any of the above

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Church's Type Theory:

- Simply typed λ-calculus with the signature

  $\mathcal{S} := \{\neg, \vee\} \cup \{\Pi^\alpha \mid \alpha \in \mathcal{T}\}$

  (and perhaps a description or choice operator).

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HOL: Other Fragments

$\mathcal{S}$ Fragment of Elementary Type Theory:

$\mathcal{S}$ Fragment of Extensional Type Theory:

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HOL: Other Fragments

$\mathcal{S}$ Fragment of Elementary Type Theory:

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$\mathcal{S}$ Fragment of Extensional Type Theory:

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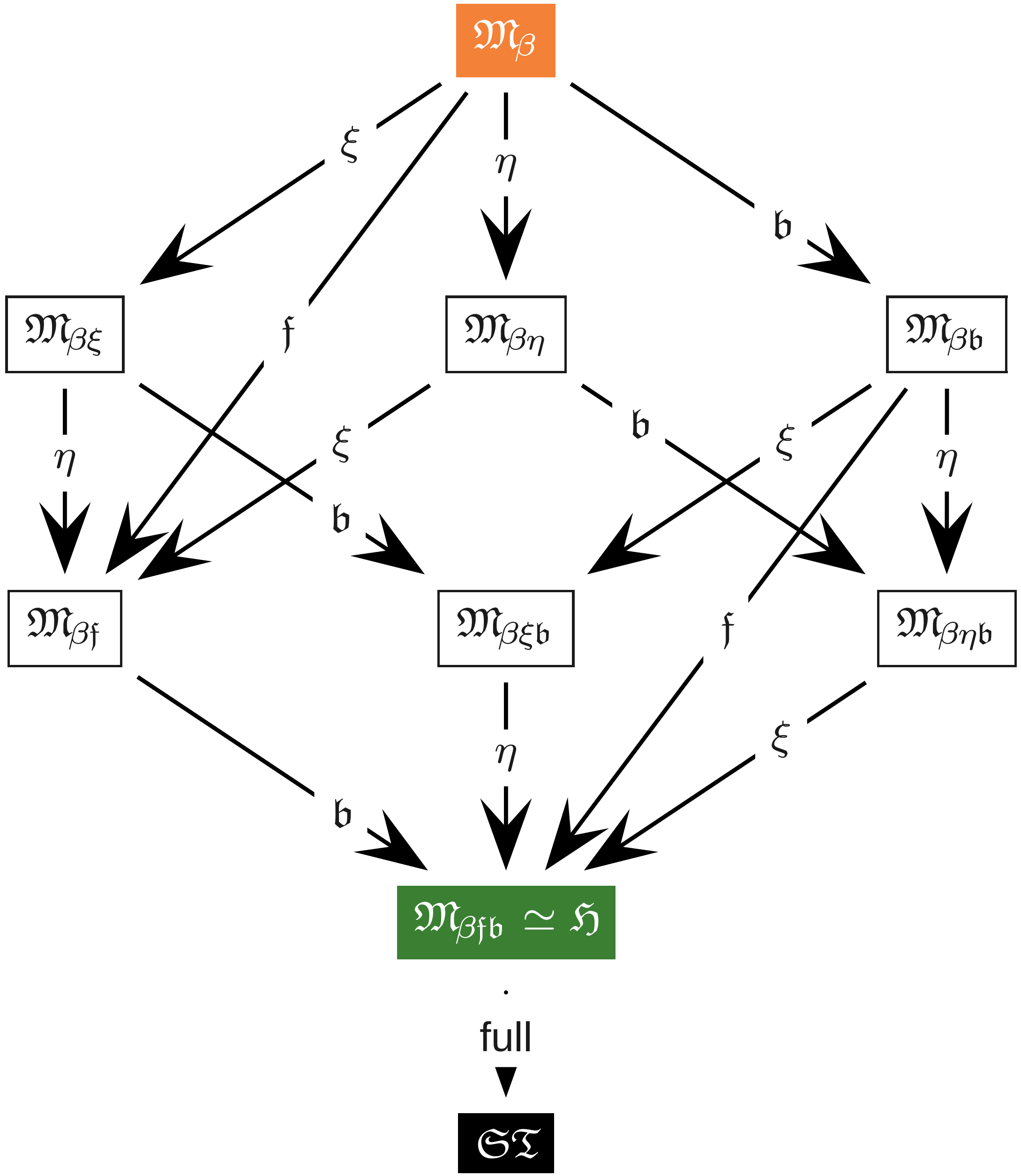

Landscape of HOL model classes

[Kohlhase-PhD-94]

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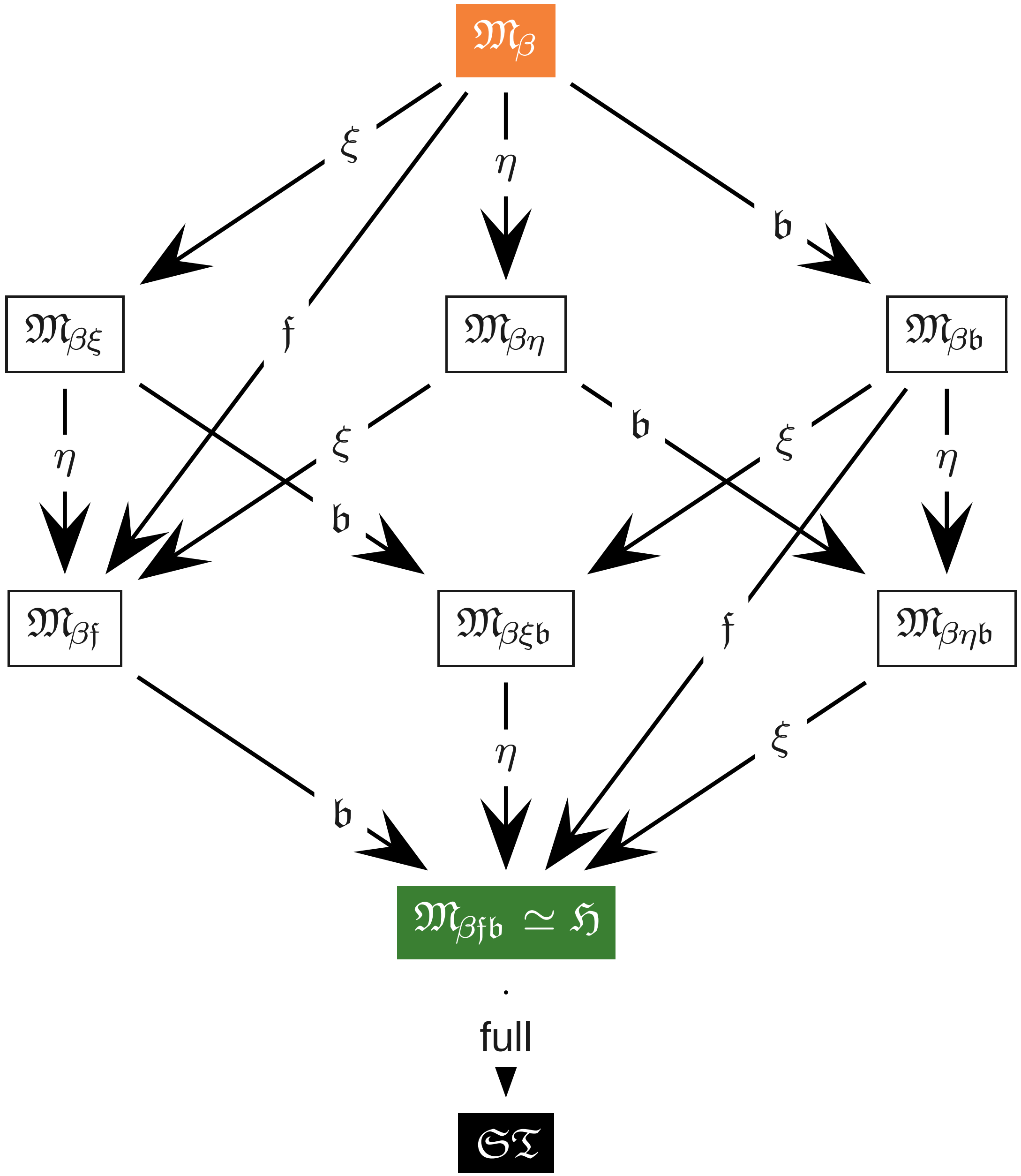

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$\mathfrak{M}_\beta$ model class for $\mathcal{S}$ fragment of elementary type theory

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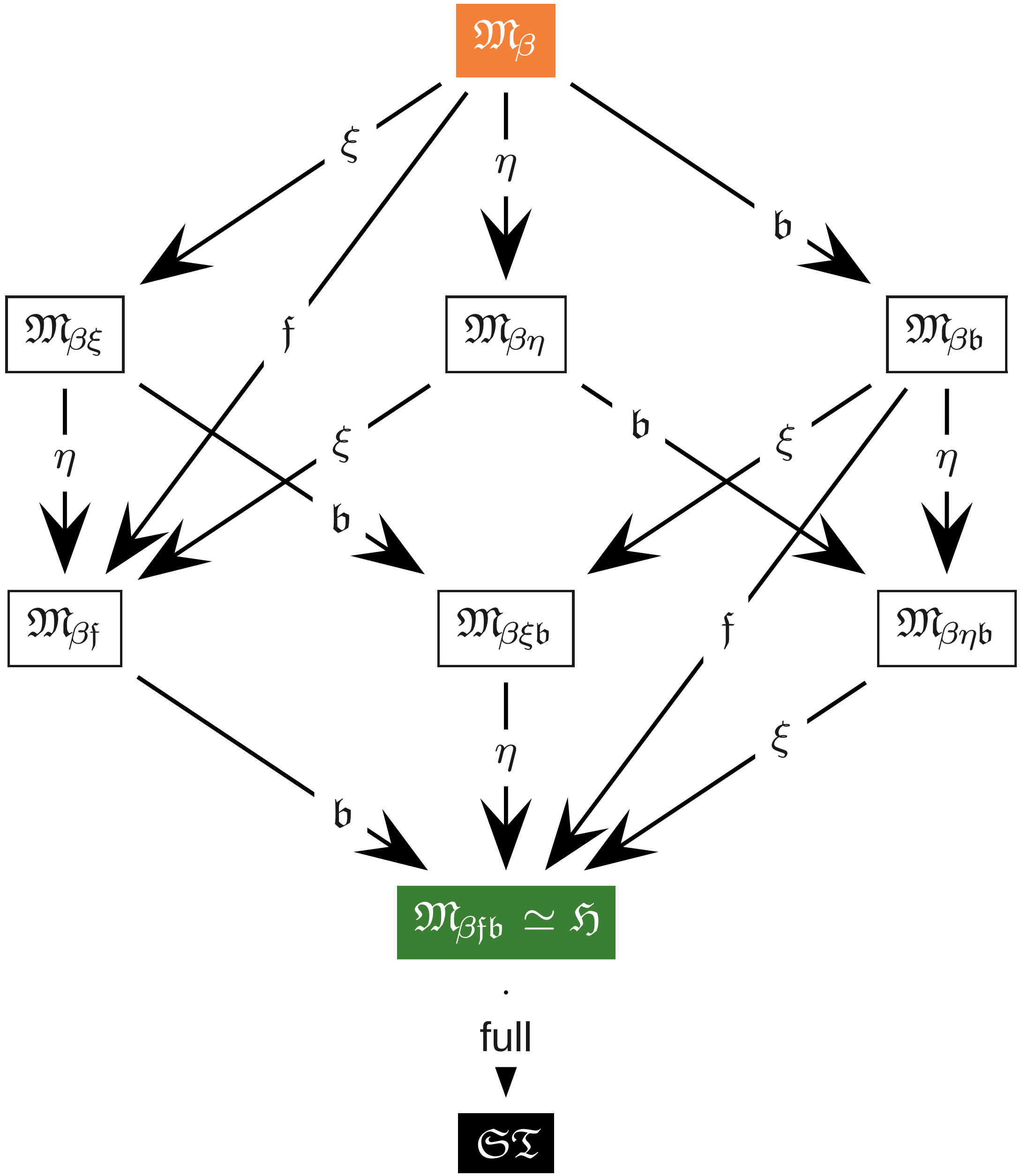

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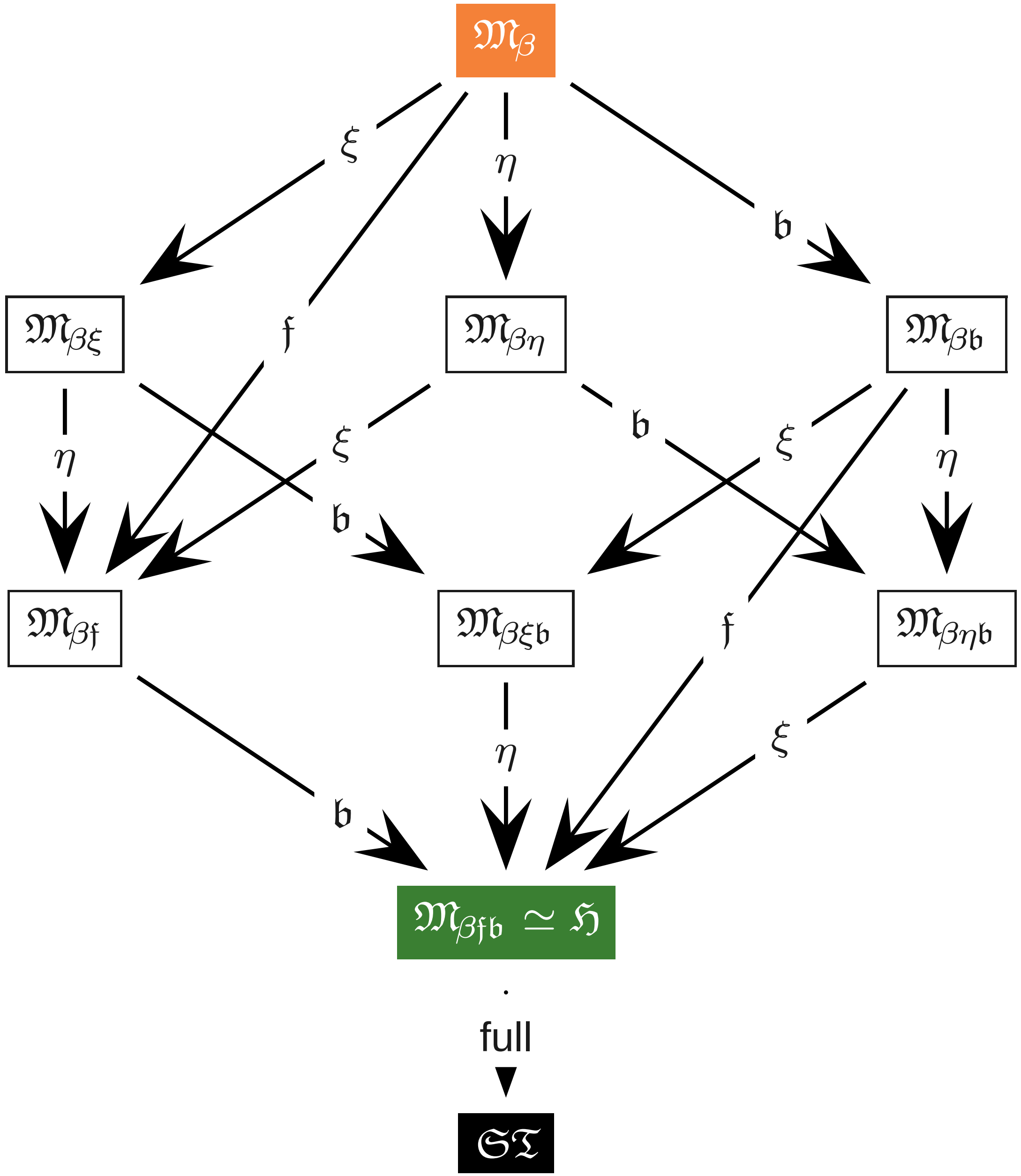

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Signature $\mathcal{S}$ defined as

$\{\top, \bot, \neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\} \cup \{\Pi^\alpha, \Sigma^\alpha, =^\alpha\}$

(less logical connectives are possible)

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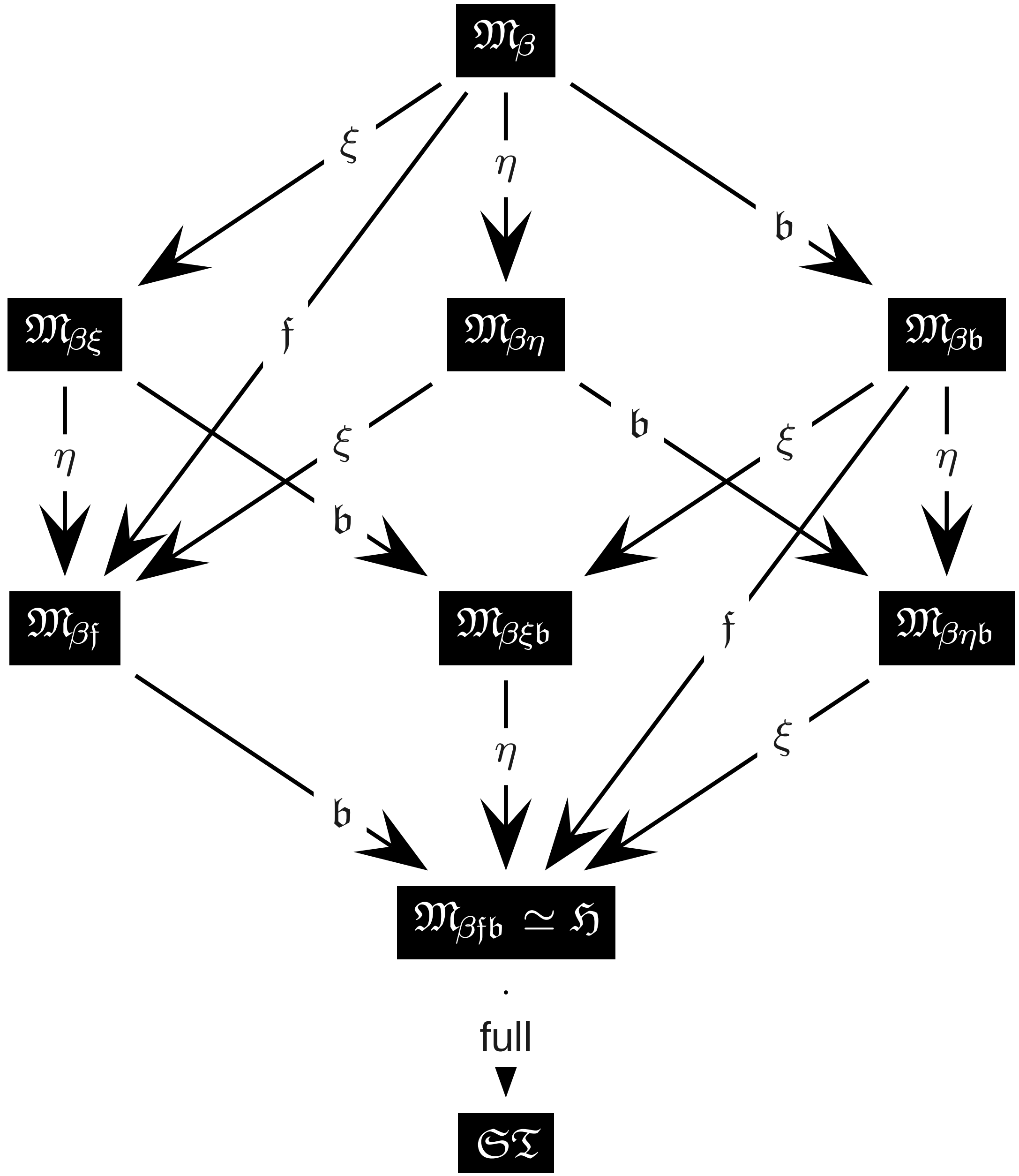

β: models support β-equality

$\mathfrak{q}$: models provide identity relations

$\forall\alpha : \mathsf{id} \in \mathcal{D}_{\alpha\to\alpha\to o}$

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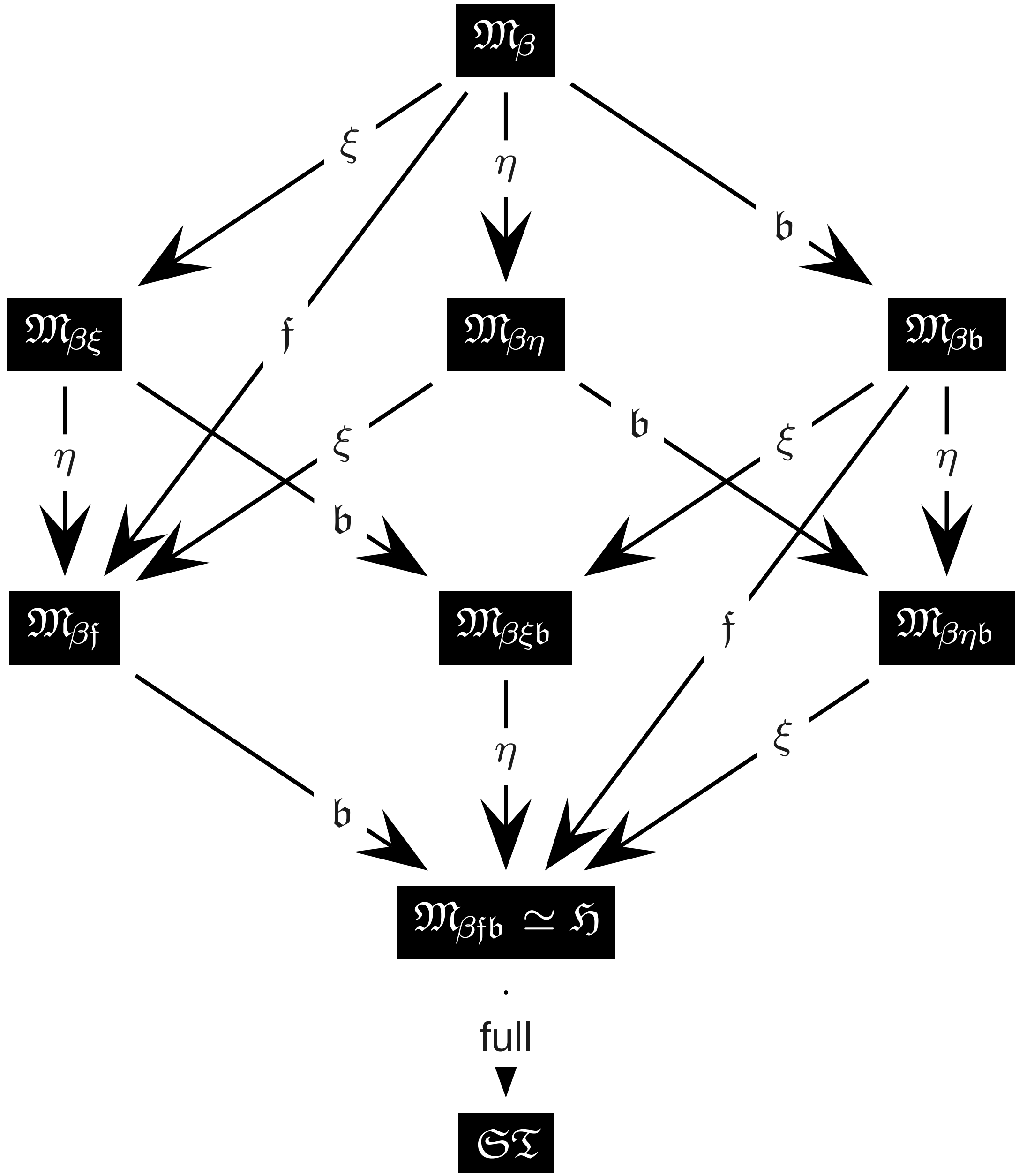

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$$\forall\alpha : \mathsf{id} \in \mathcal{D}_{\alpha\to\alpha\to o}$$

- [Andrews72]: without property $\mathfrak{q}$ Leibniz equality $\dot{=}$ not necessarily evaluates to identity relation — even in Henkin semantics

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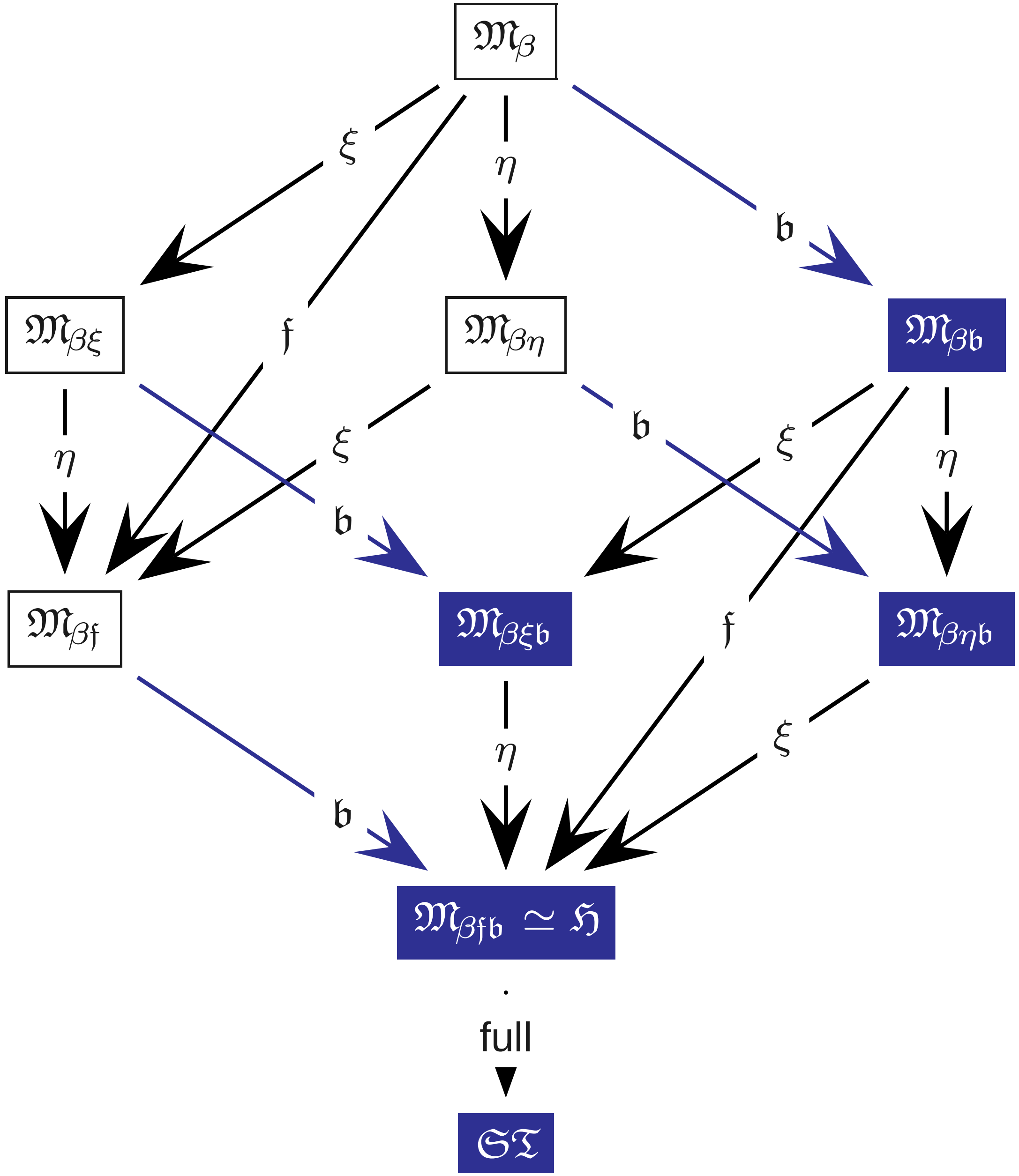

$\mathfrak{b}$: models are Boolean extensional

$$\mathcal{D}_o \equiv \{\bot, \top\}$$

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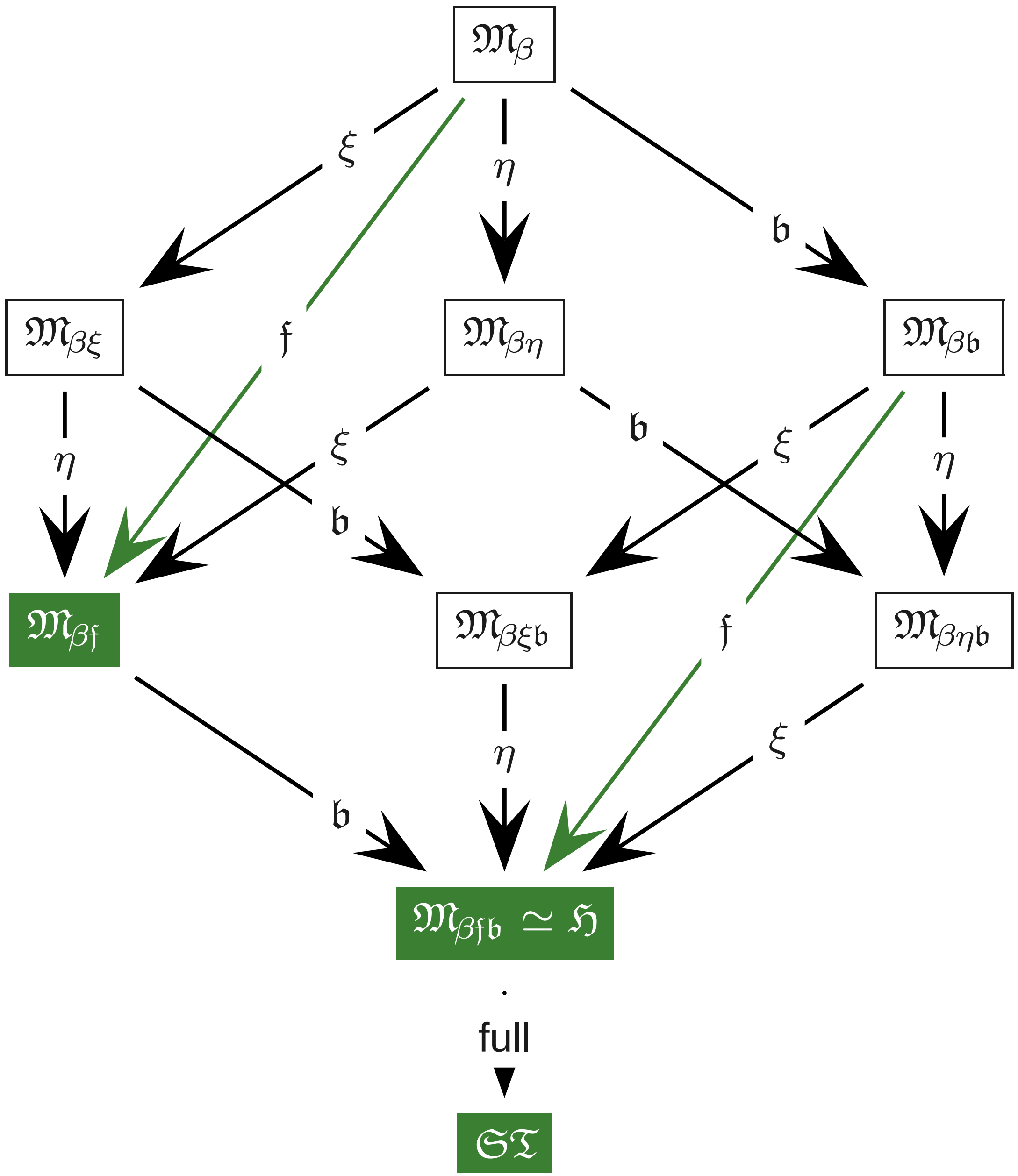

$\mathfrak{f}$: models are functional

$\forall f, g \in \mathcal{D}_{\beta\alpha}$:
$f \equiv g$ iff $f@a \equiv g@a$ $(\forall a \in \mathcal{D}_\alpha)$

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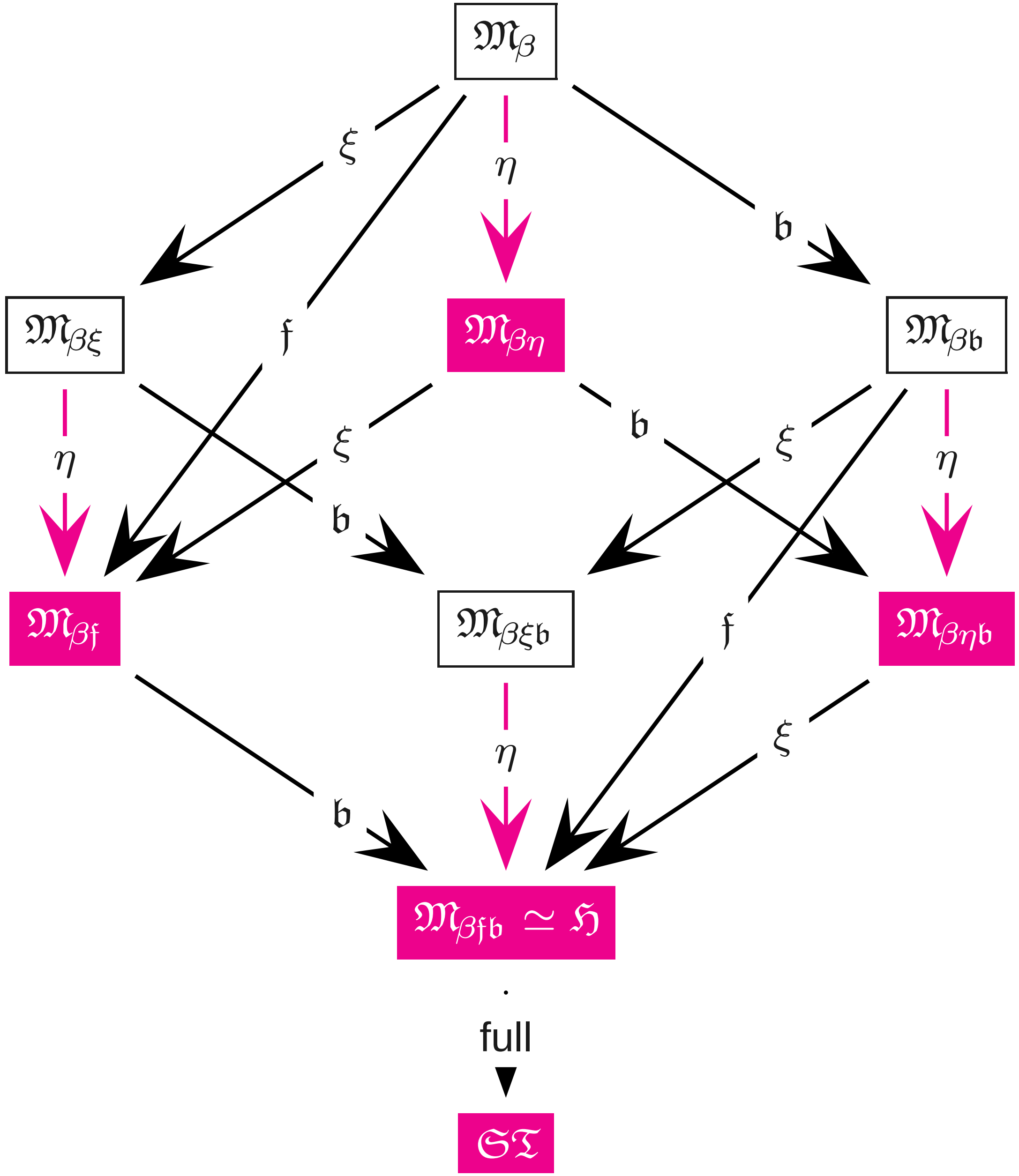

η: models are η-functional

$$\mathcal{E}_\varphi(\mathsf{A}) \equiv \mathcal{E}_\varphi(\mathsf{A}\downarrow_\eta)$$

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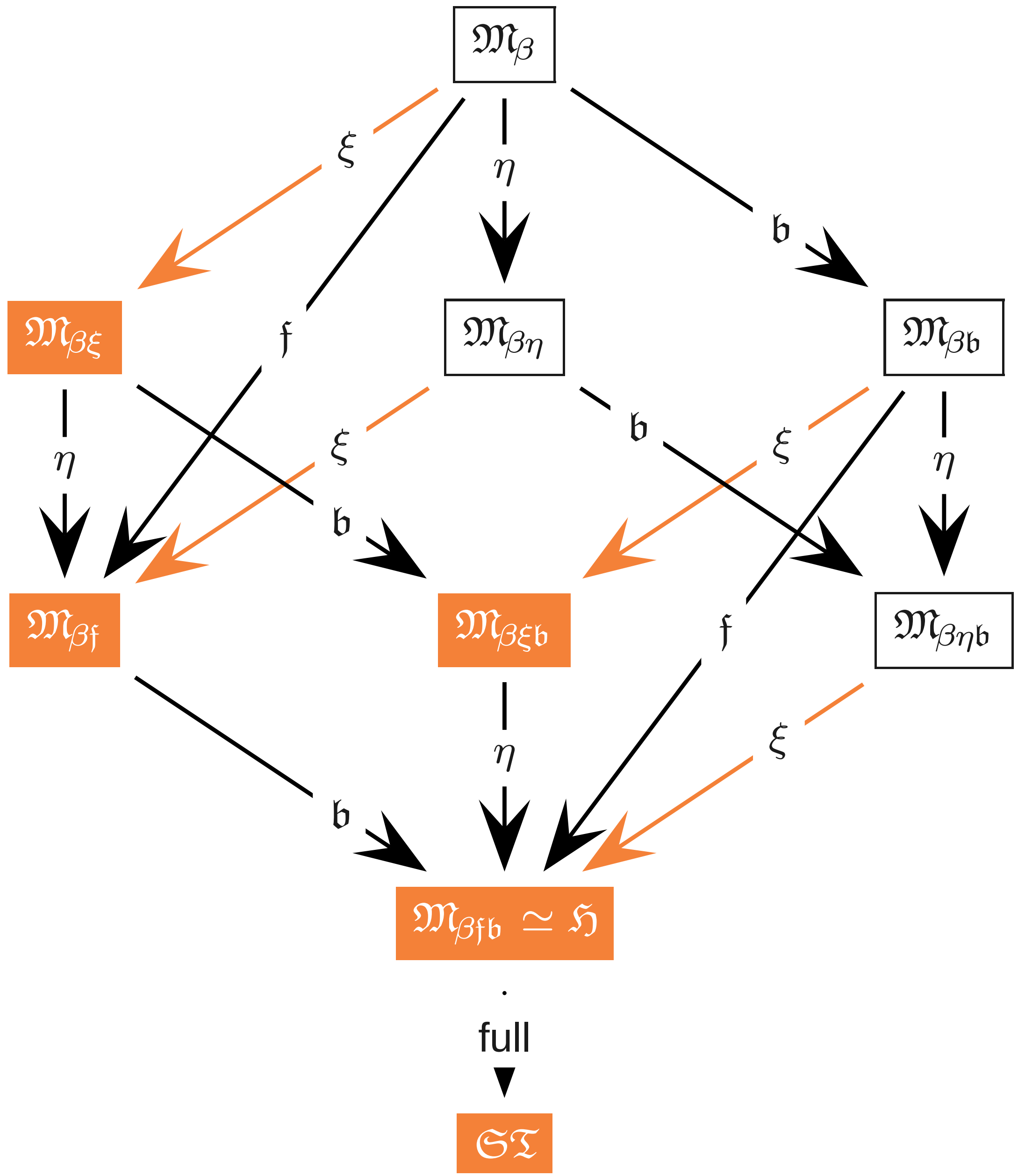

ξ: models are ξ-functional

$$\mathcal{E}_\varphi(\lambda X_\alpha.\, M_\beta) \equiv \mathcal{E}_\varphi(\lambda X_\alpha.\, N_\beta) \text{ iff}$$
$$\mathcal{E}_{\varphi,[a/X]}(M) \equiv \mathcal{E}_{\varphi,[a/X]}(N)\ (\forall a \in \mathcal{D}_\alpha)$$

HOL-CUBE: Abstract Consistency

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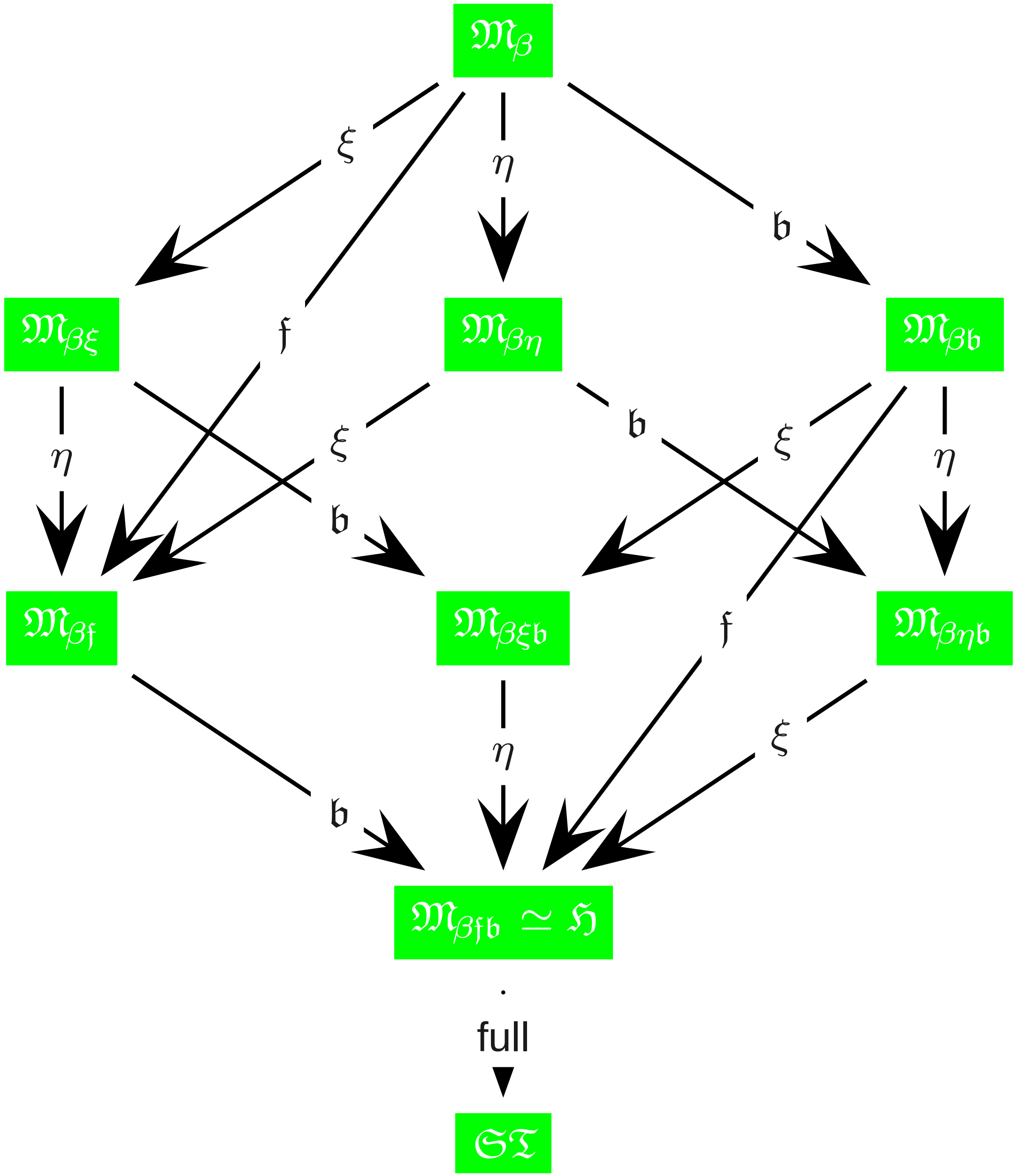
- They support completeness and soundness analysis of calculi by syntactical means for the HOL-CUBE
- Proposal:
  use the examples of this paper before trying a formal analysis

HOL-Problems requiring β



Church numerals:

$\overline{\mathsf{n}}^{\alpha} := (\lambda F_{\alpha\alpha} \lambda Y_{\alpha}.\ (F^{n}\ Y))$

$\overline{+} := \lambda M \lambda N \lambda F \lambda Y.\ MF(NFY)$

$\overline{\times} := \lambda M \lambda N \lambda F \lambda Z.\ N(MF)Z$

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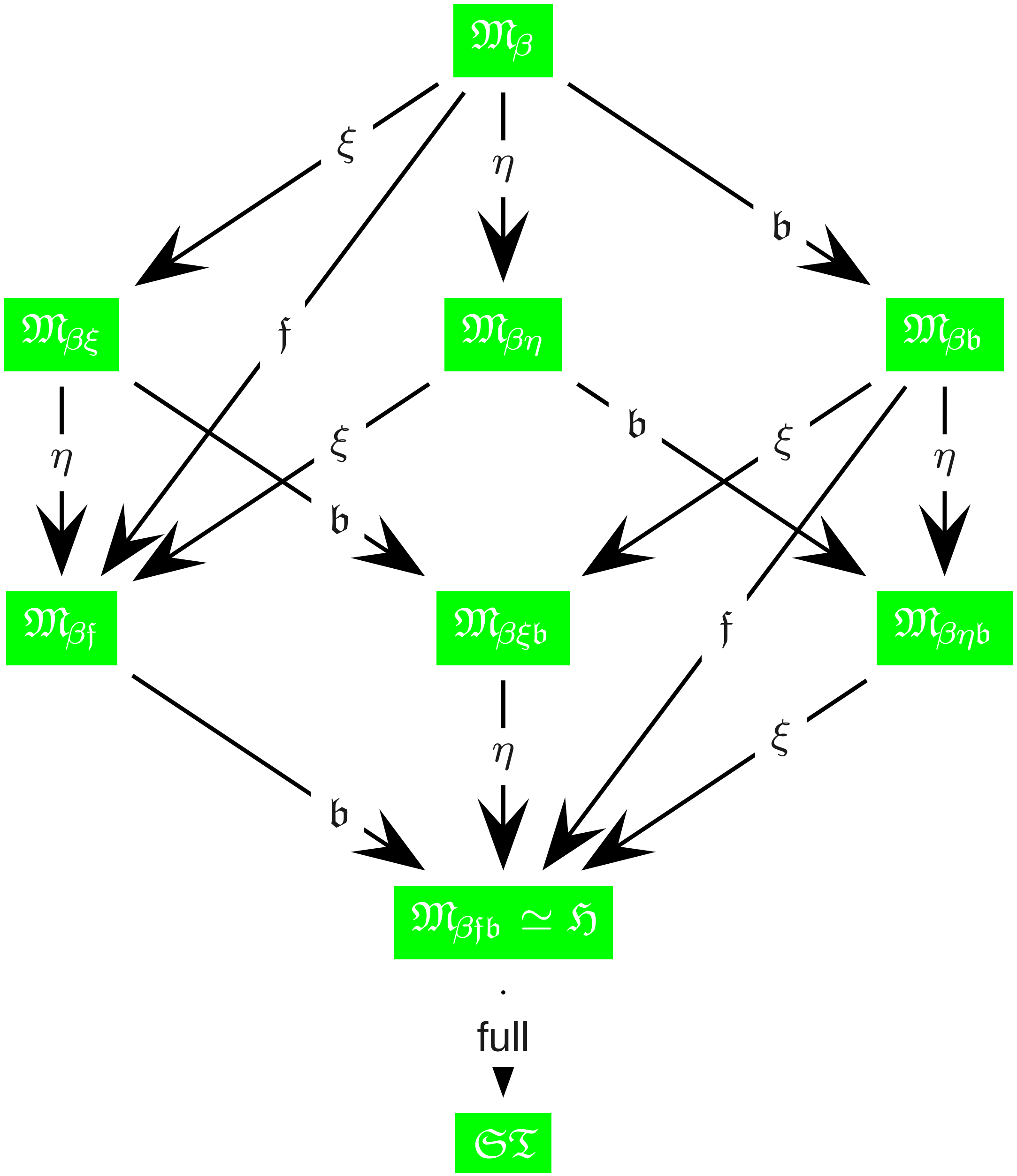

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Efficiency of β-conversion:

- $\overline{3}\overline{\times}\overline{4} \stackrel{*}{=} \overline{5}\overline{+}\overline{7}$
- $(\overline{10}\overline{\times}\overline{10})\overline{\times}\overline{10} \stackrel{*}{=} ((\overline{10}\overline{\times}\overline{5})\overline{+}(\overline{5}\overline{\times}\overline{10}))\overline{\times}\overline{10}$

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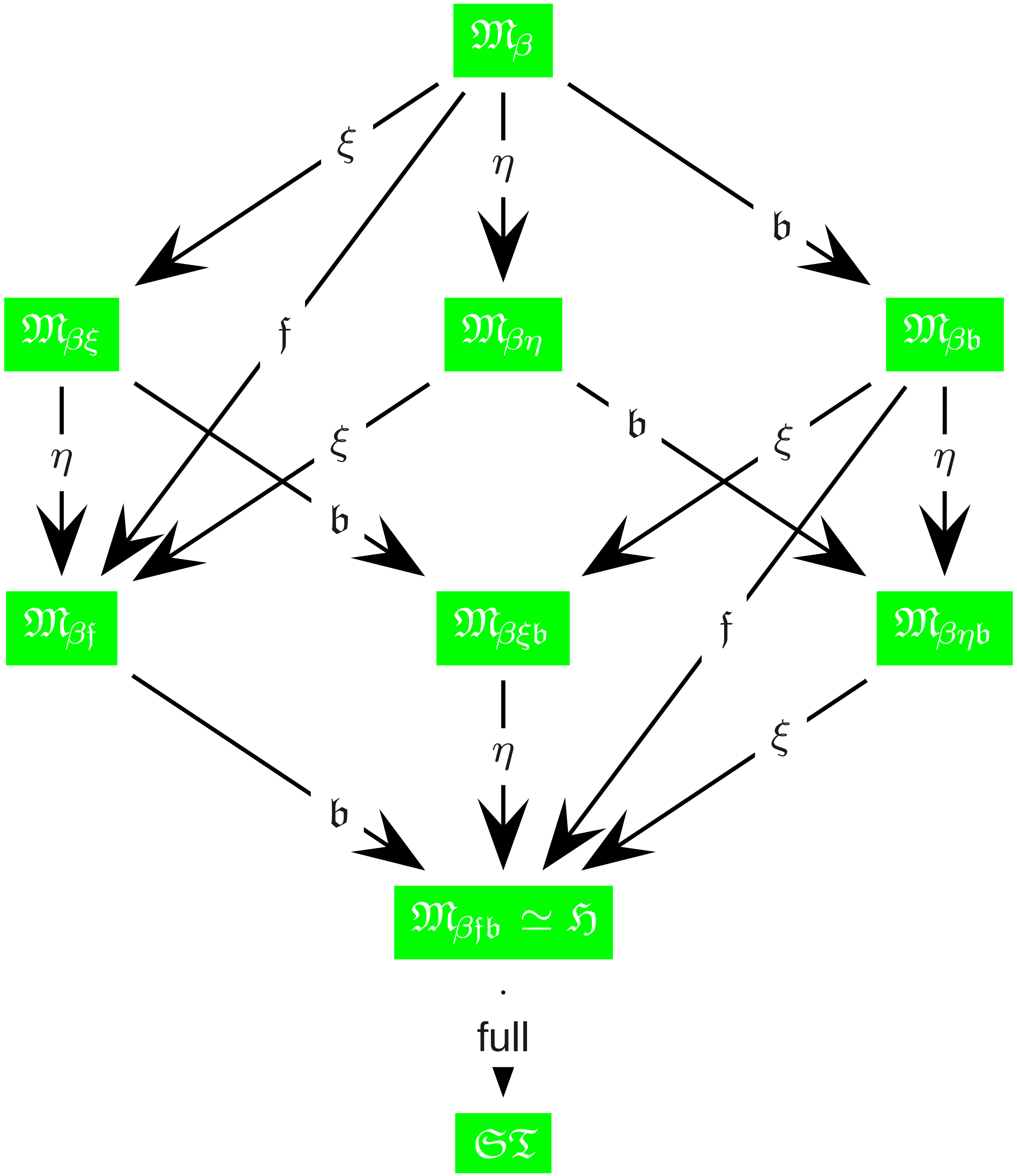

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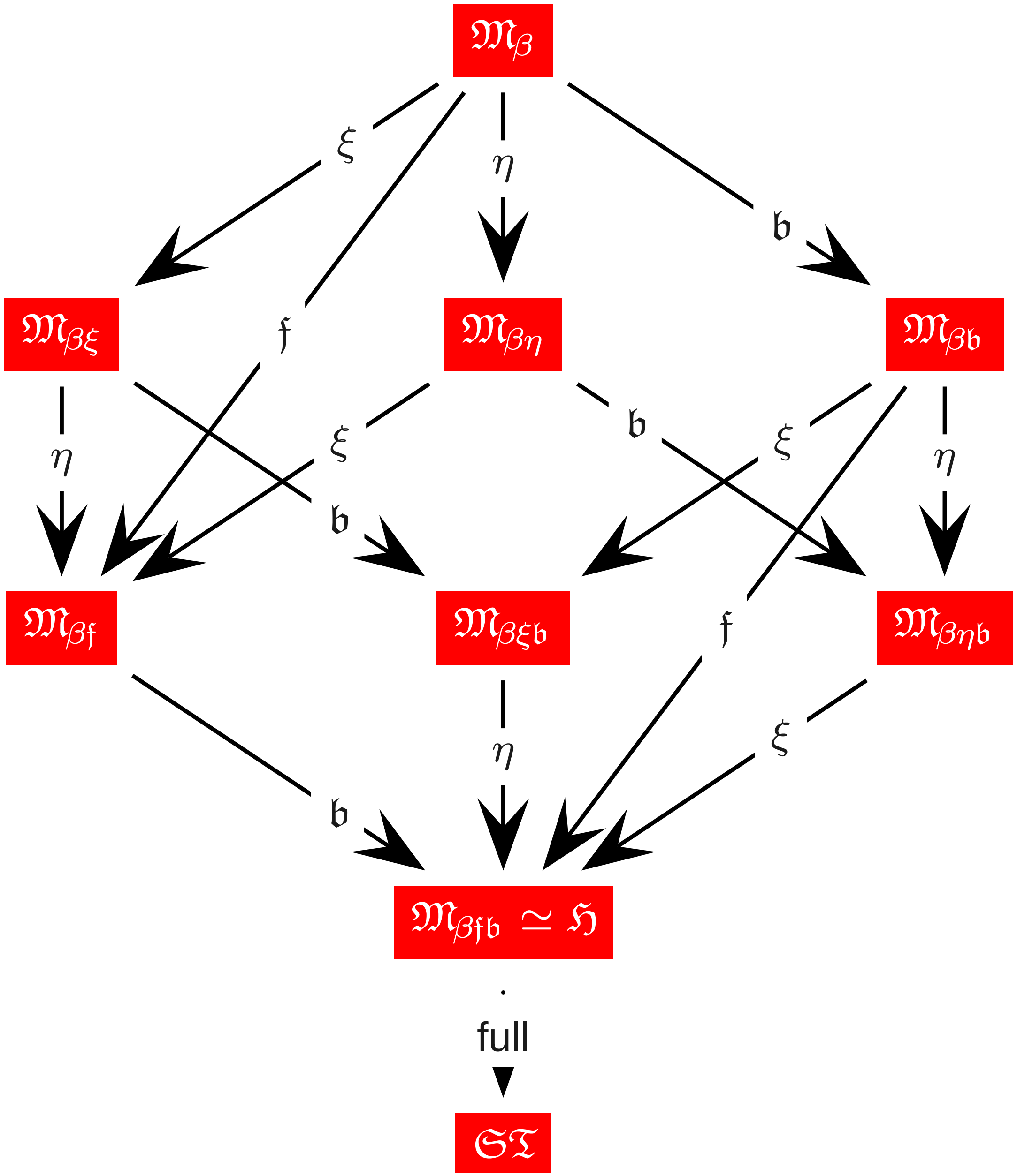
Pre-unification with β-conversion:

- $\exists N_{(\iota\to\iota)\to\iota\to\iota}.\, ((N\overline{\times}\overline{1}) \stackrel{*}{=} \overline{1})$
  (two solutions if only β;
  one solution if $\beta\eta$)
- $\exists N.\, N\overline{\times}\overline{4} \stackrel{*}{=} \overline{5}\overline{+}\overline{7}$
- $\exists H.\, ((H\ \overline{2})\overline{3}) \stackrel{*}{=} \overline{6} \wedge ((H\ \overline{1})\overline{2}) \stackrel{*}{=} \overline{2}$
- $\exists N, M.\, N\overline{\times}\overline{4} \stackrel{*}{=} \overline{5}\overline{+}M$
  (infinitely many solutions!)

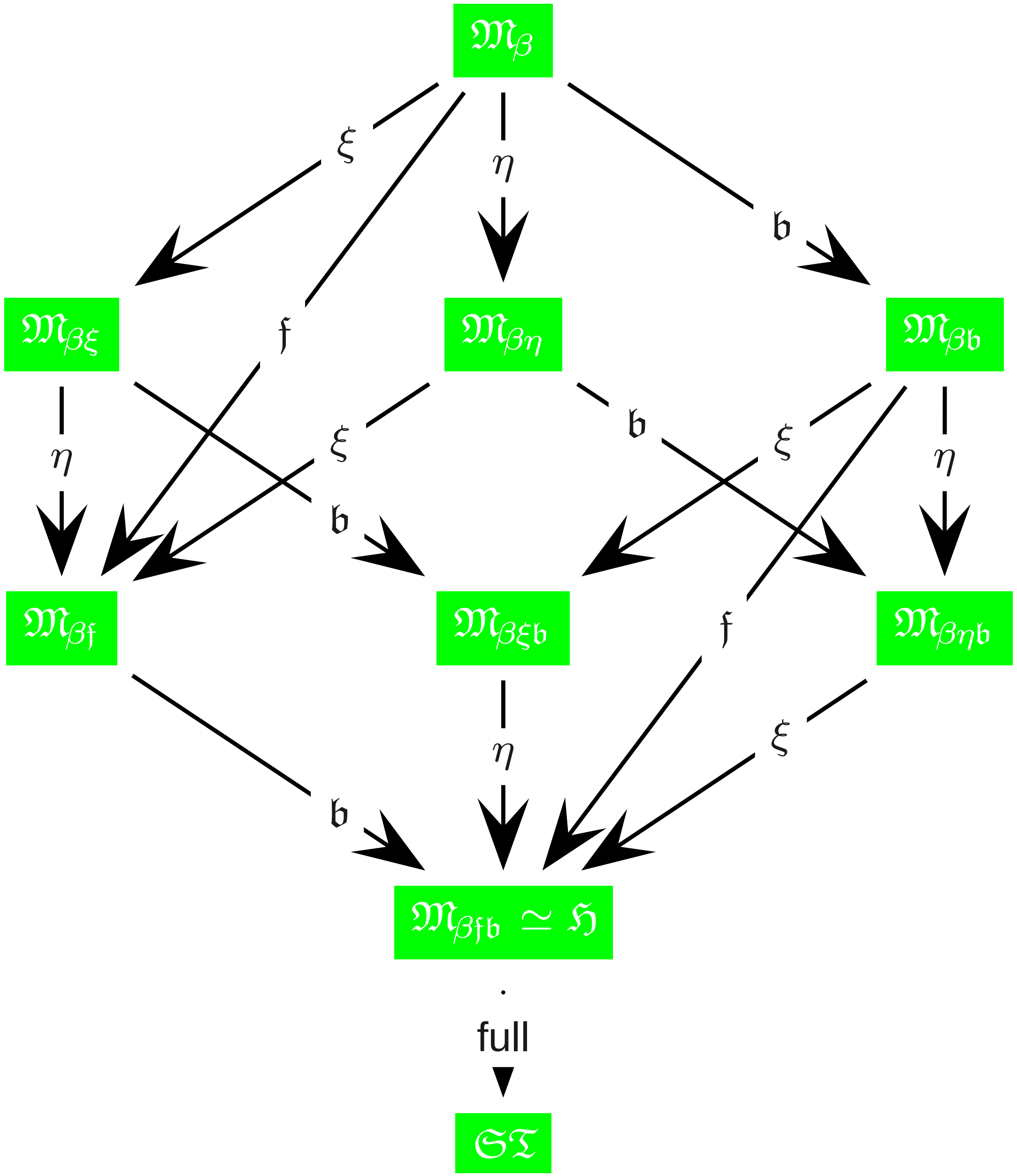
HOL-Non-Problems



Some non-theorems:

- essentially FOL
- apply to all model classes
- address
  - Skolemization
  - axiom of choice

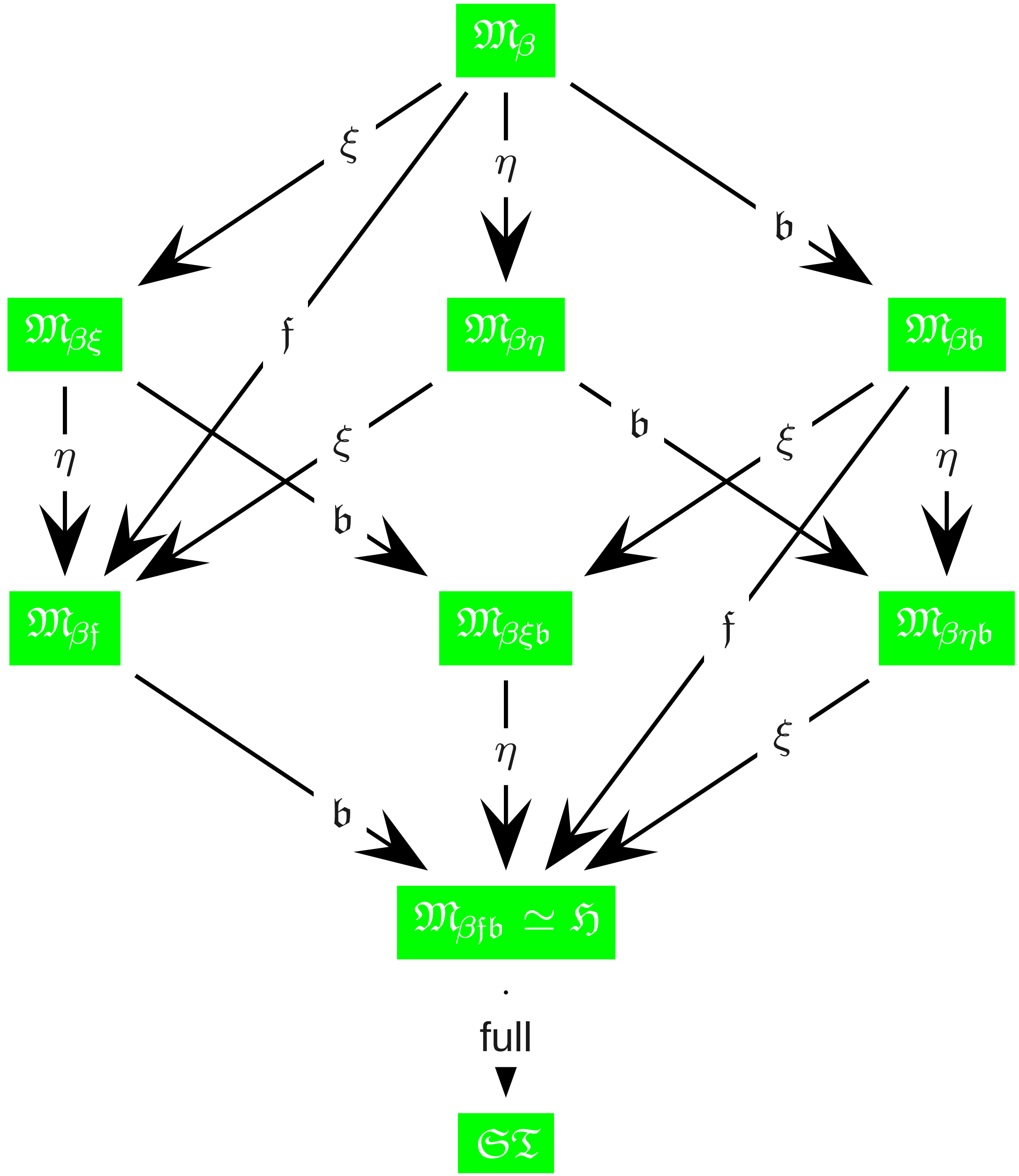
HOL-Problems requiring β



$\stackrel{*}{\doteq}$ is equivalence relation

- $\forall X_\alpha.\, X \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha.\, X \stackrel{*}{\doteq} Y \Rightarrow Y \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha.\, (X \stackrel{*}{\doteq} Y \wedge Y \stackrel{*}{\doteq} Z) \Rightarrow X \stackrel{*}{\doteq} Z$

HOL-Problems requiring β



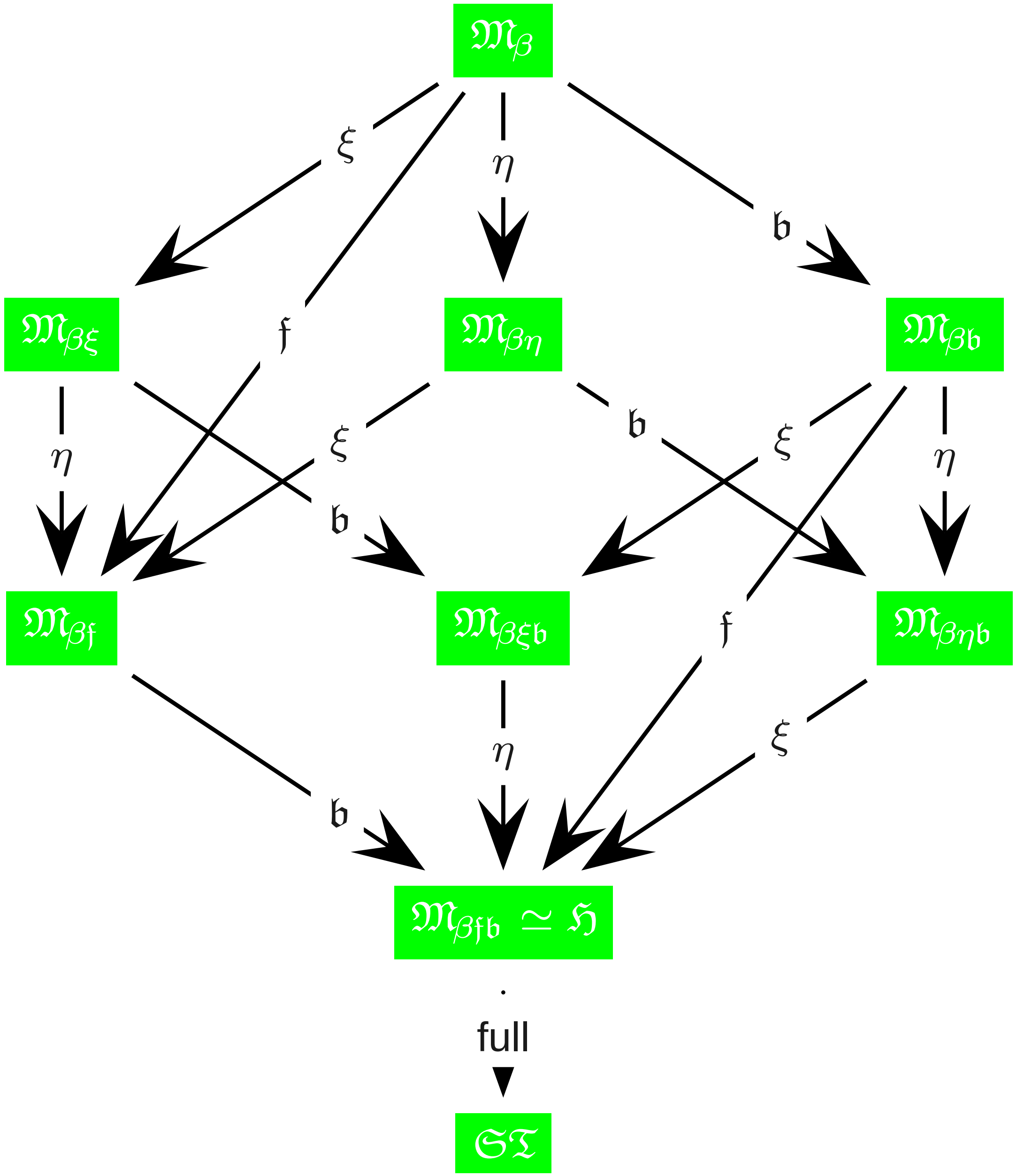
$\stackrel{*}{\doteq}$ is equivalence relation

- $\forall X_\alpha \centerdot X \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha \centerdot X \stackrel{*}{\doteq} Y \Rightarrow Y \stackrel{*}{\doteq} X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha \centerdot (X \stackrel{*}{\doteq} Y \wedge Y \stackrel{*}{\doteq} Z) \Rightarrow X \stackrel{*}{\doteq} Z$

$\stackrel{*}{\doteq}$ is congruence relation

- $\forall X_\alpha, Y_\alpha, F_{\alpha\alpha} \centerdot X \stackrel{*}{\doteq} Y \Rightarrow (FX) \stackrel{*}{\doteq} (FY)$
- $\forall X_\alpha, Y_\alpha, P_{o\alpha} \centerdot X \stackrel{*}{\doteq} Y \wedge (PX) \Rightarrow (PY)$

HOL-Problems requiring β



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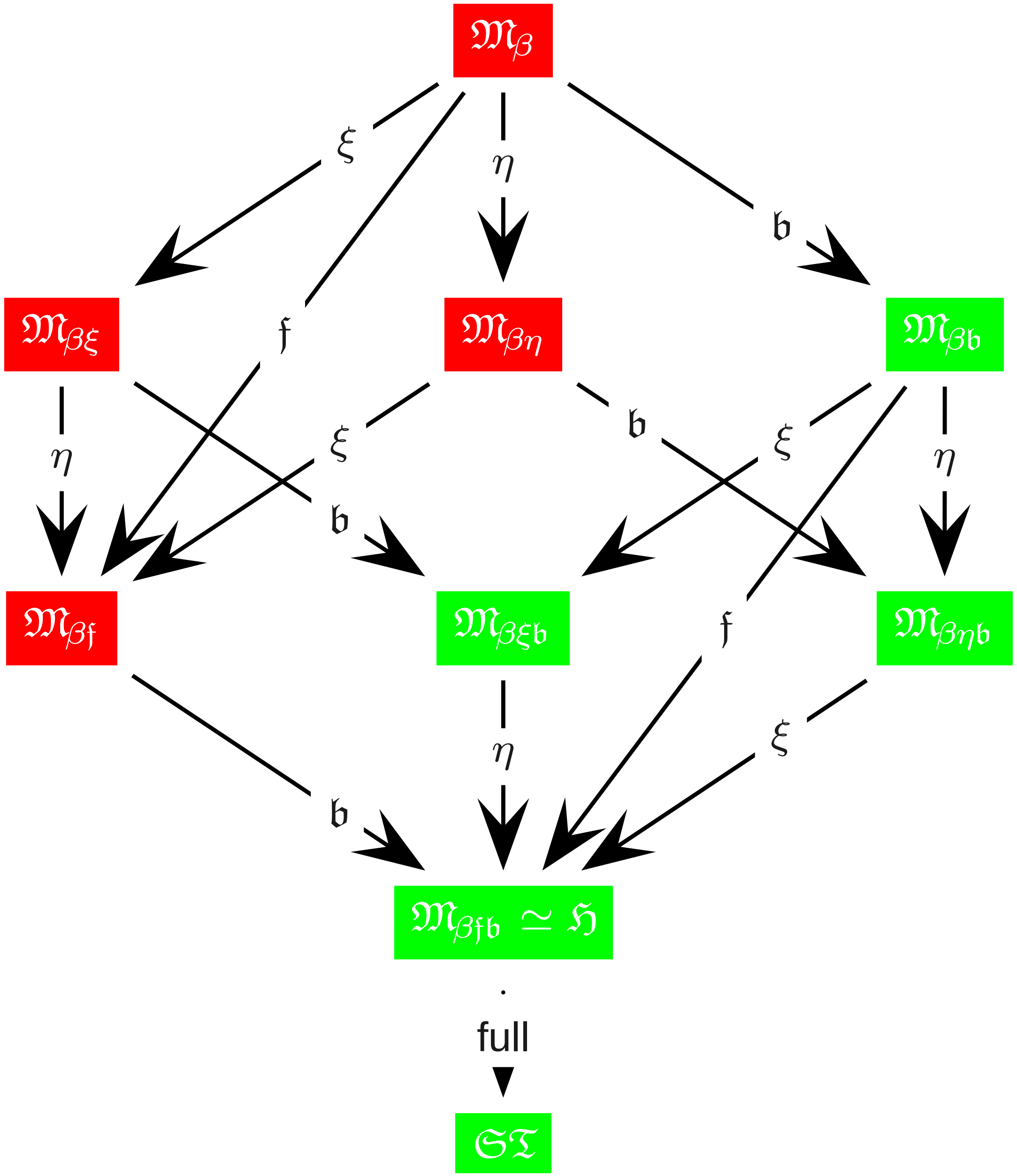
Trivial directions of Boolean and functional extensionality

- $\forall A_o, B_o.\, A \stackrel{*}{\doteq} B \Rightarrow (A \Leftrightarrow B)$
- $\forall F_{\beta\alpha}, G_{\beta\alpha}.\, F \stackrel{*}{\doteq} G \Rightarrow (\forall X_\alpha.\, FX \stackrel{*}{\doteq} GX)$

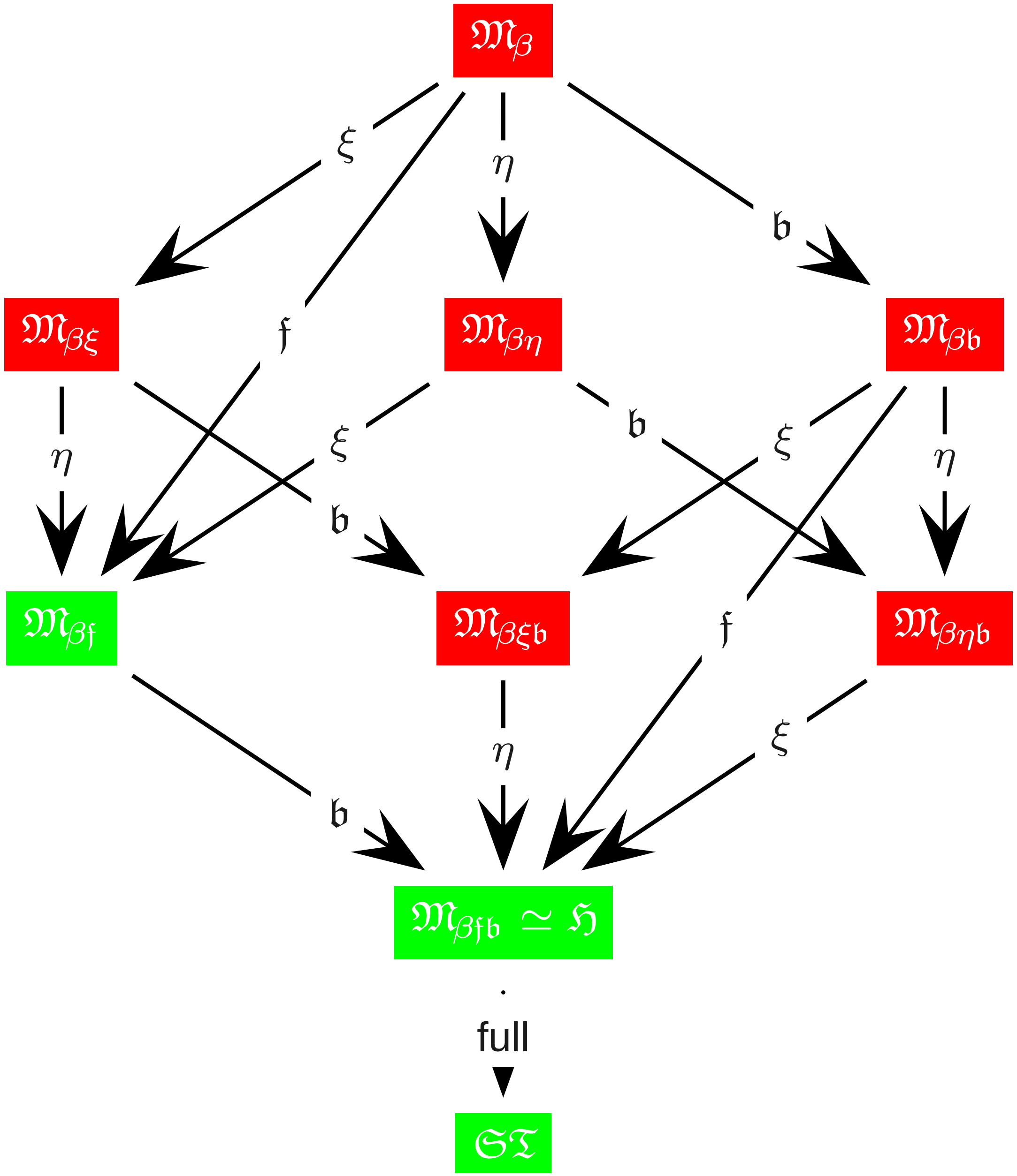
HOL-Problems requiring $\mathfrak{b}$



Non-trivial direction of Boolean extensionality

- $\forall A_o, B_o.\ (A \Leftrightarrow B) \Rightarrow A \stackrel{*}{=} B$

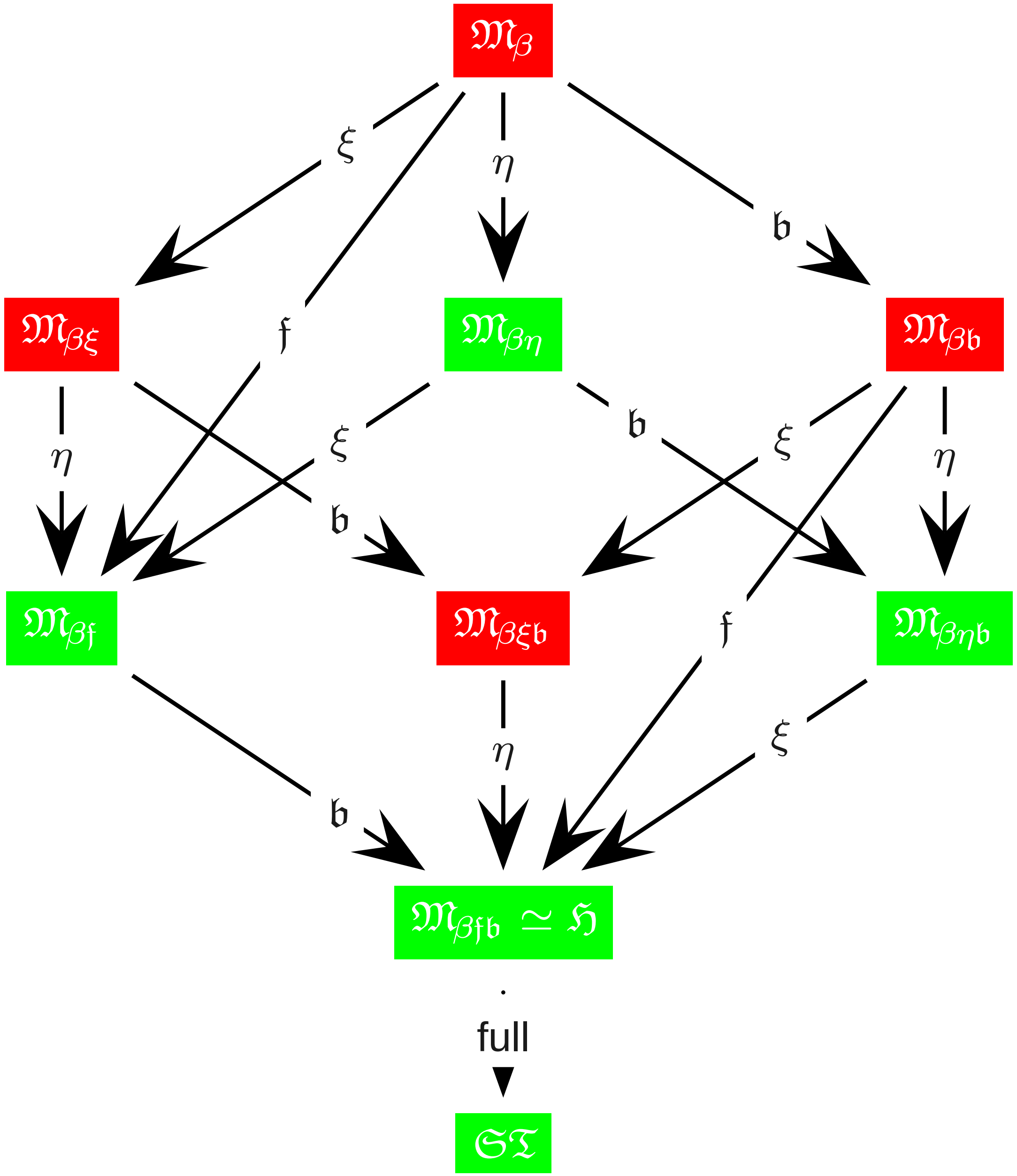
HOL-Problems requiring $\mathfrak{f}$



Non-trivial direct. of functional extensionality

- $\forall F_{\beta\alpha}, G_{\beta\alpha}. (\forall X_\alpha. FX \stackrel{*}{=} GX) \Rightarrow F \stackrel{*}{=} G$

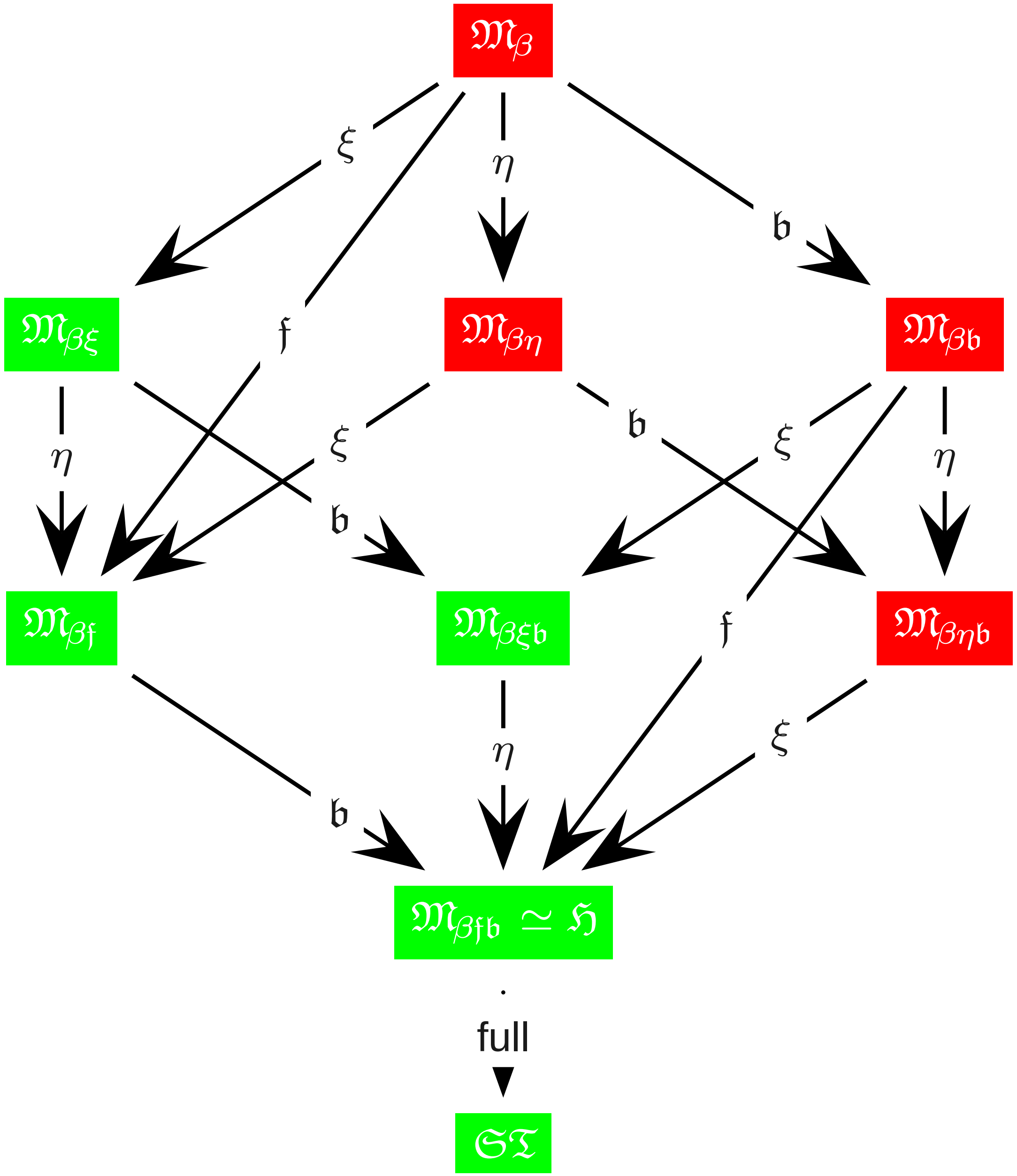
HOL-Problems requiring η



Example requiring property η

- $(p_{o(\iota\iota)}(\lambda X_\iota.\, f_{\iota\iota}X)) \Rightarrow (p\ f)$

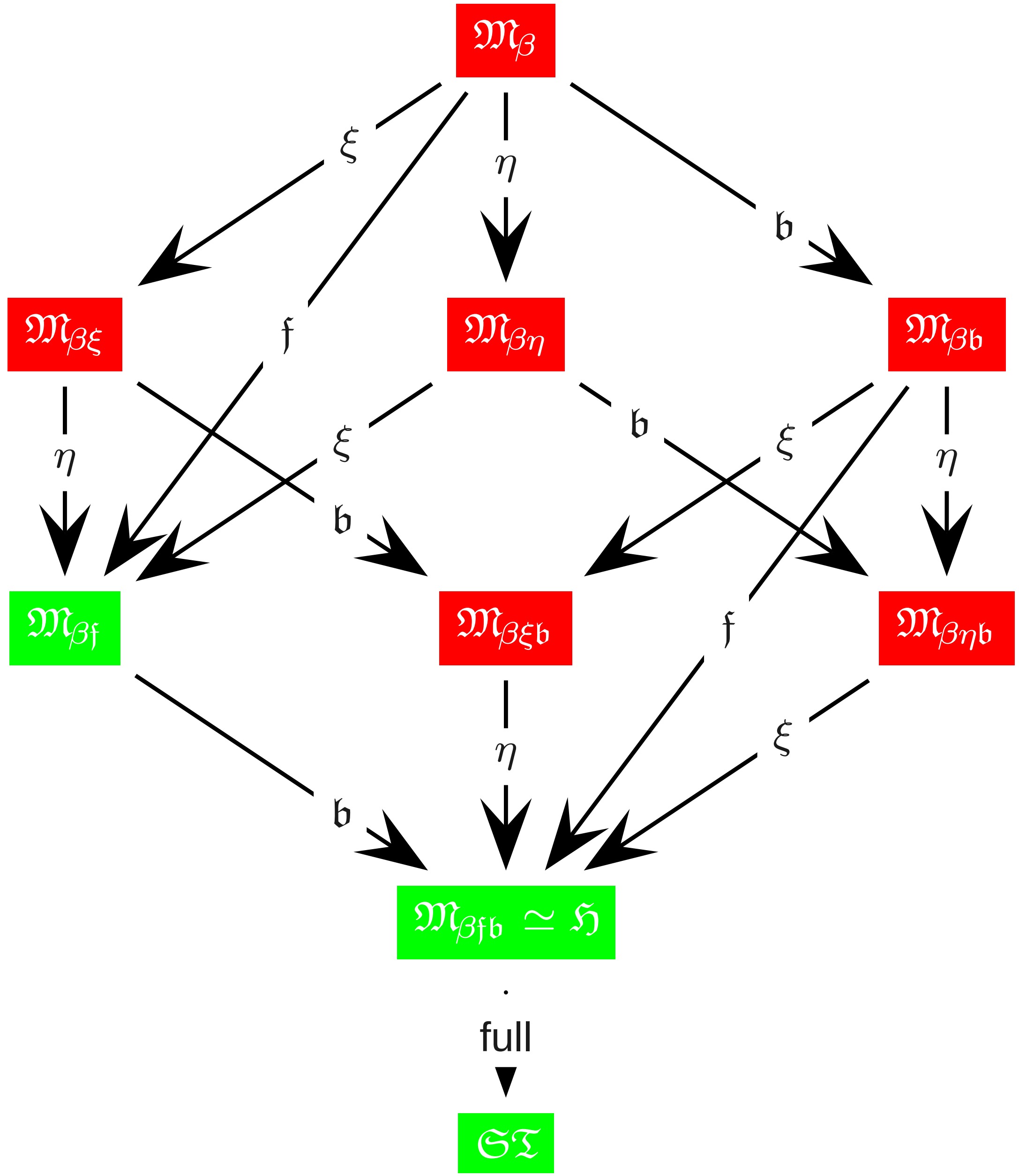
HOL-Problems requiring ξ



Example requiring property ξ (and $\mathfrak{q}$!)

- $(\forall X_\iota . (f_{\iota\iota} X) \stackrel{*}{\doteq} X) \land p_{o(\iota\iota)}(\lambda X_\iota . X)$
  $\Rightarrow p(\lambda X_\iota . fX)$

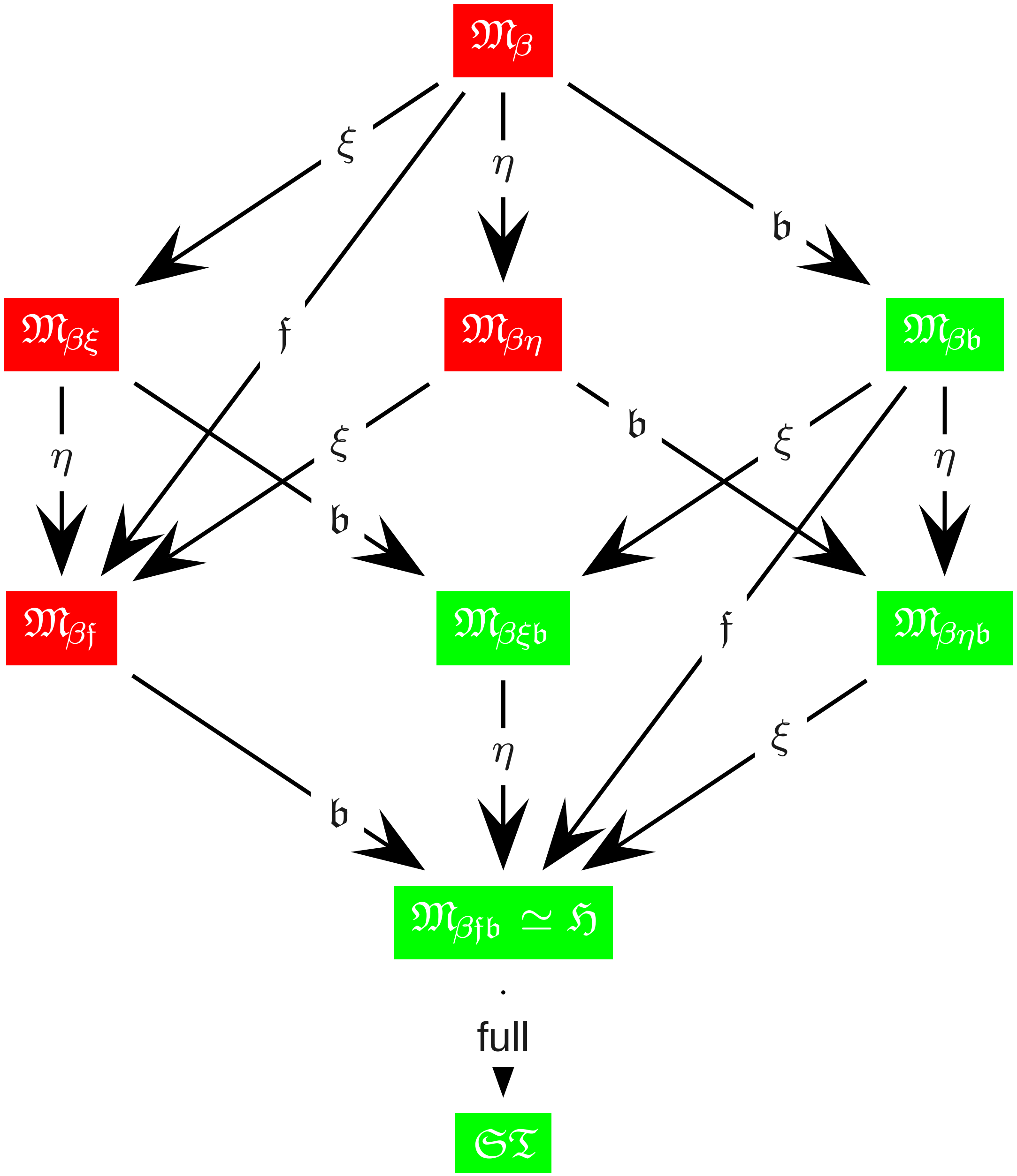
HOL-Problems requiring $\mathfrak{f}$



Example requiring property $\mathfrak{f}$ (and $\mathfrak{q}$!)

- $(\forall X_\iota.\, (f_{\iota\iota} X) \stackrel{*}{=} X) \wedge p_{o(\iota\iota)} (\lambda X_\iota.\, X)$
  $\Rightarrow (p\ f)$

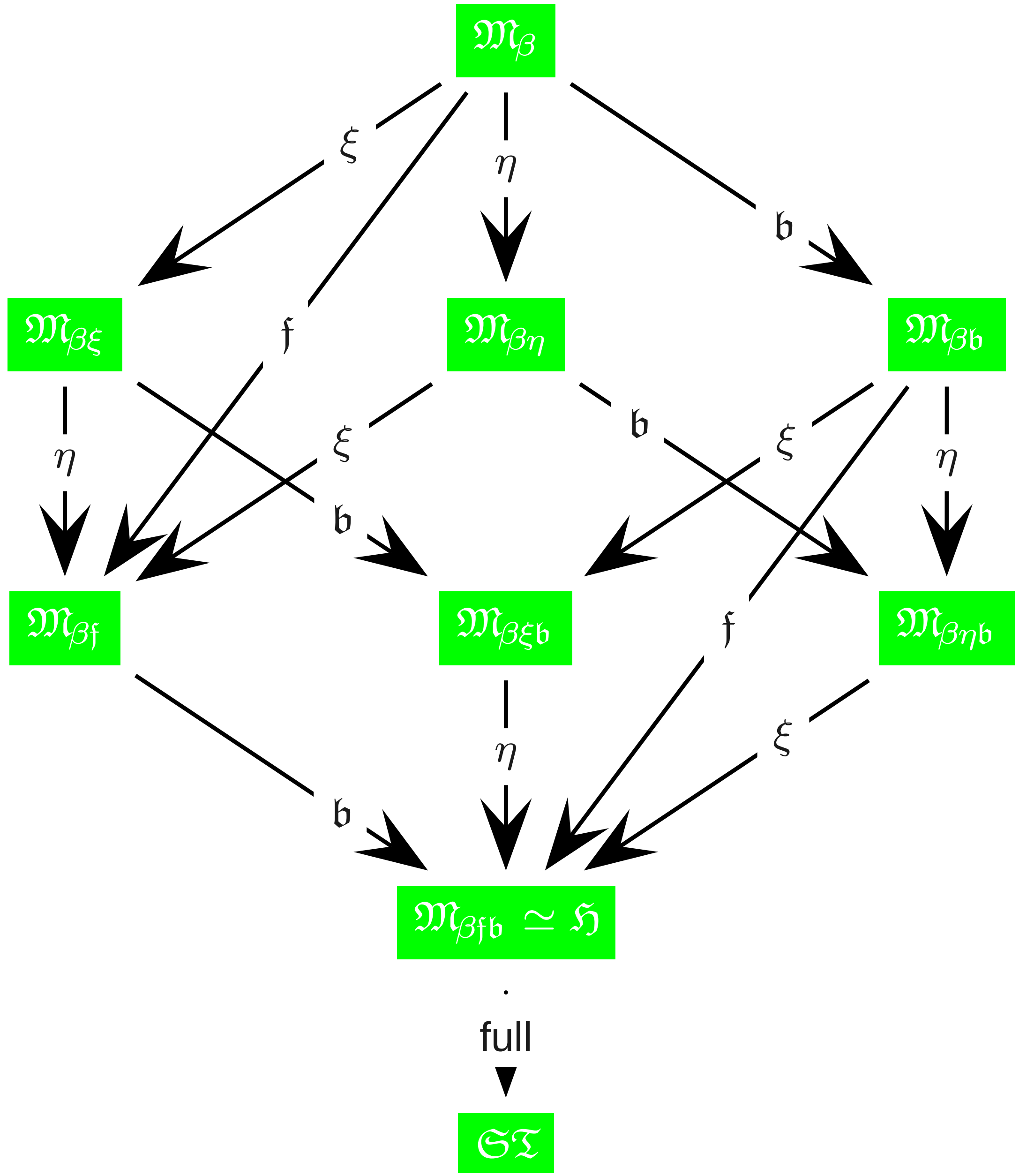
HOL-Problems requiring $\mathfrak{b}$



Examples requiring property $\mathfrak{b}$

- $(p_{oo}\ a_o) \wedge (p\ b_o) \Rightarrow (p\ (a \wedge b))$
- $\neg(a \stackrel{*}{=} \neg a)$ (in particular $\neg(a = \neg a)$)
- $(h_{\iota o}((h\top) \stackrel{*}{=} (h\bot))) \stackrel{*}{=} (h\bot)$

HOL-Problems: Other Examples

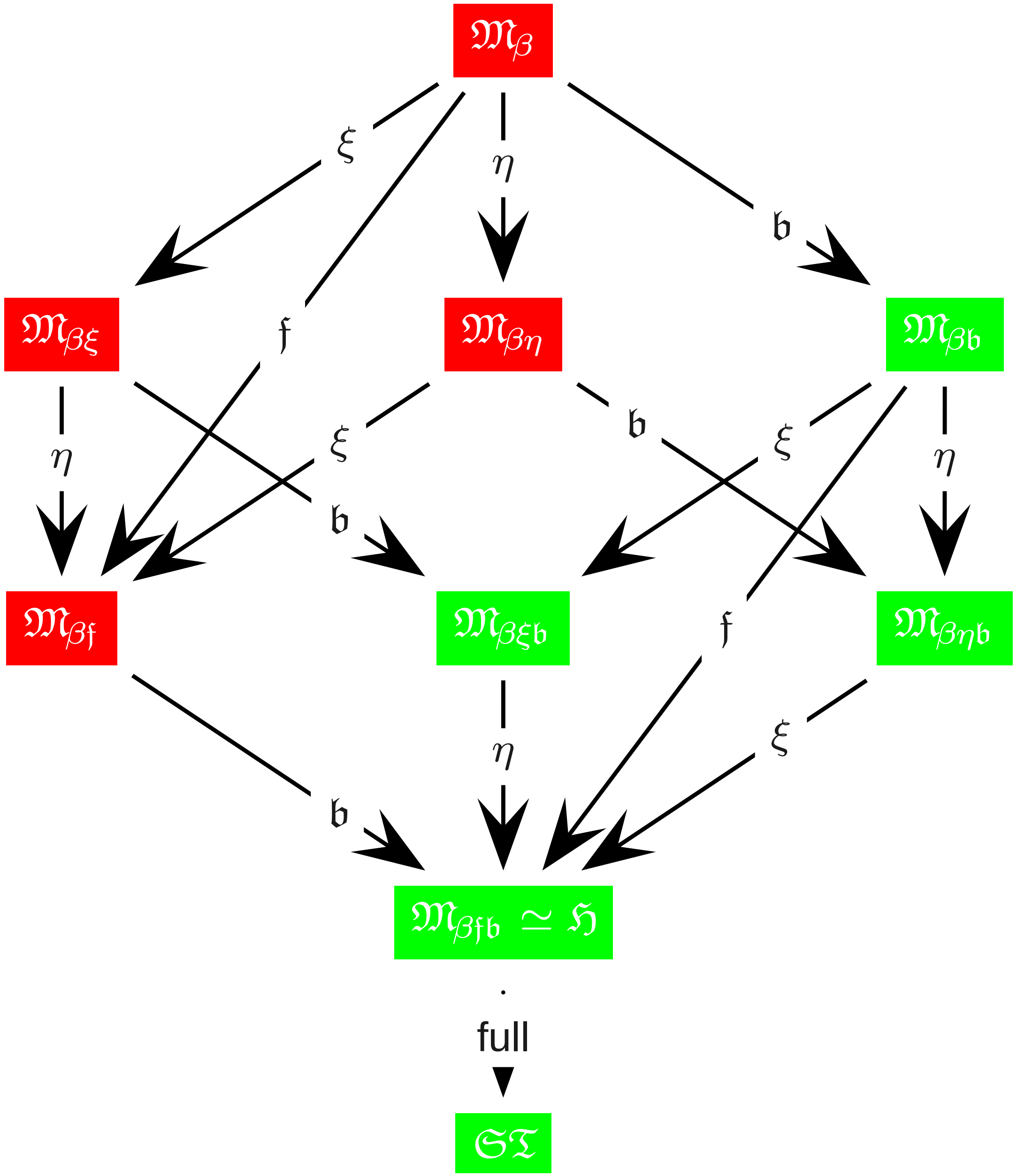


Playing with DeMorgan's Law:

- $\forall X, Y.\ X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$

'Ok' for all model classes

HOL-Problems: DeMorgan's Law

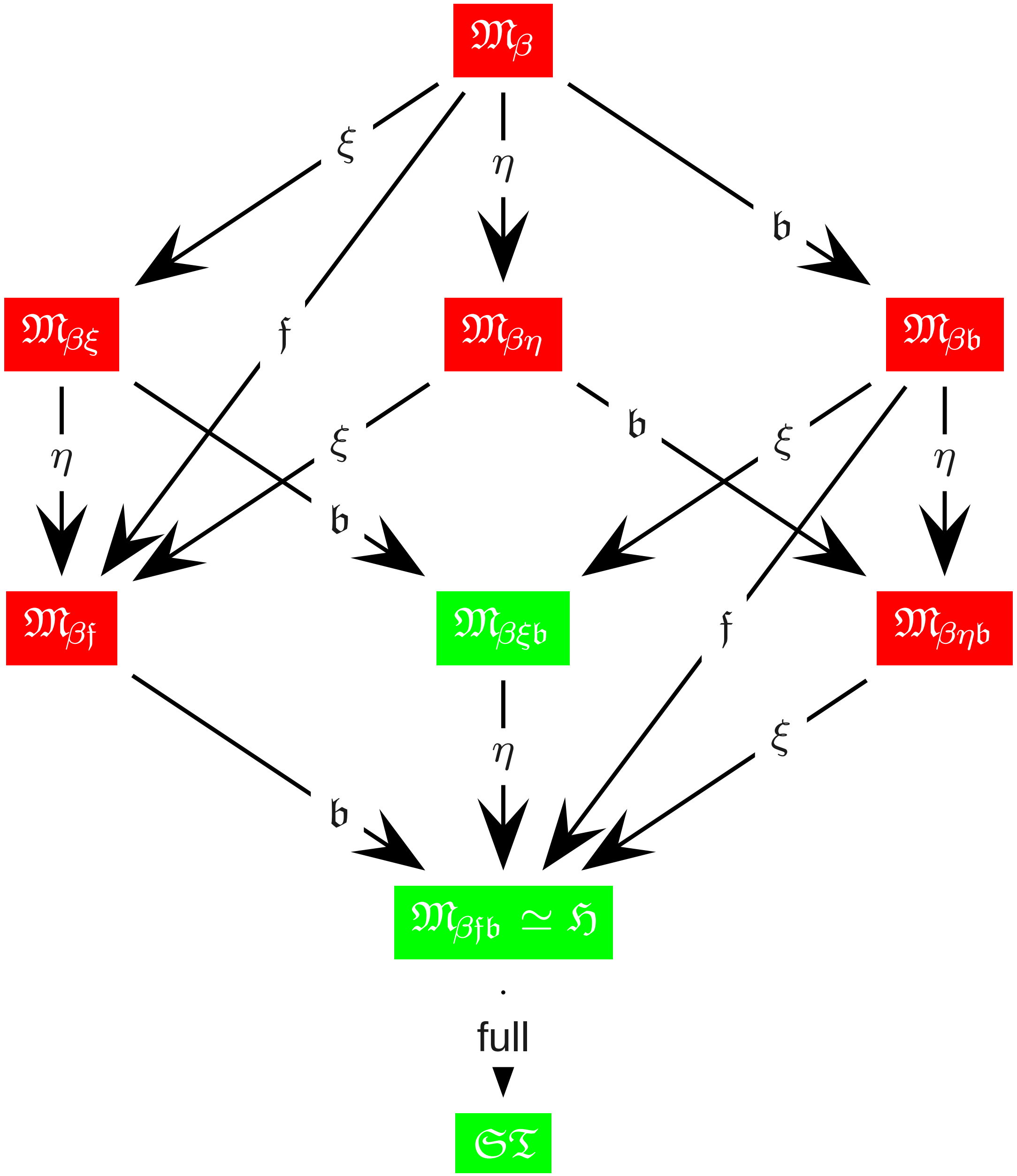


Playing with DeMorgan's Law:

- $\forall X, Y. X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$
- $\forall X, Y. X \wedge Y \stackrel{*}{=} \neg(\neg X \vee \neg Y)$

requires $\mathfrak{b}$

HOL-Problems: DeMorgan's Law

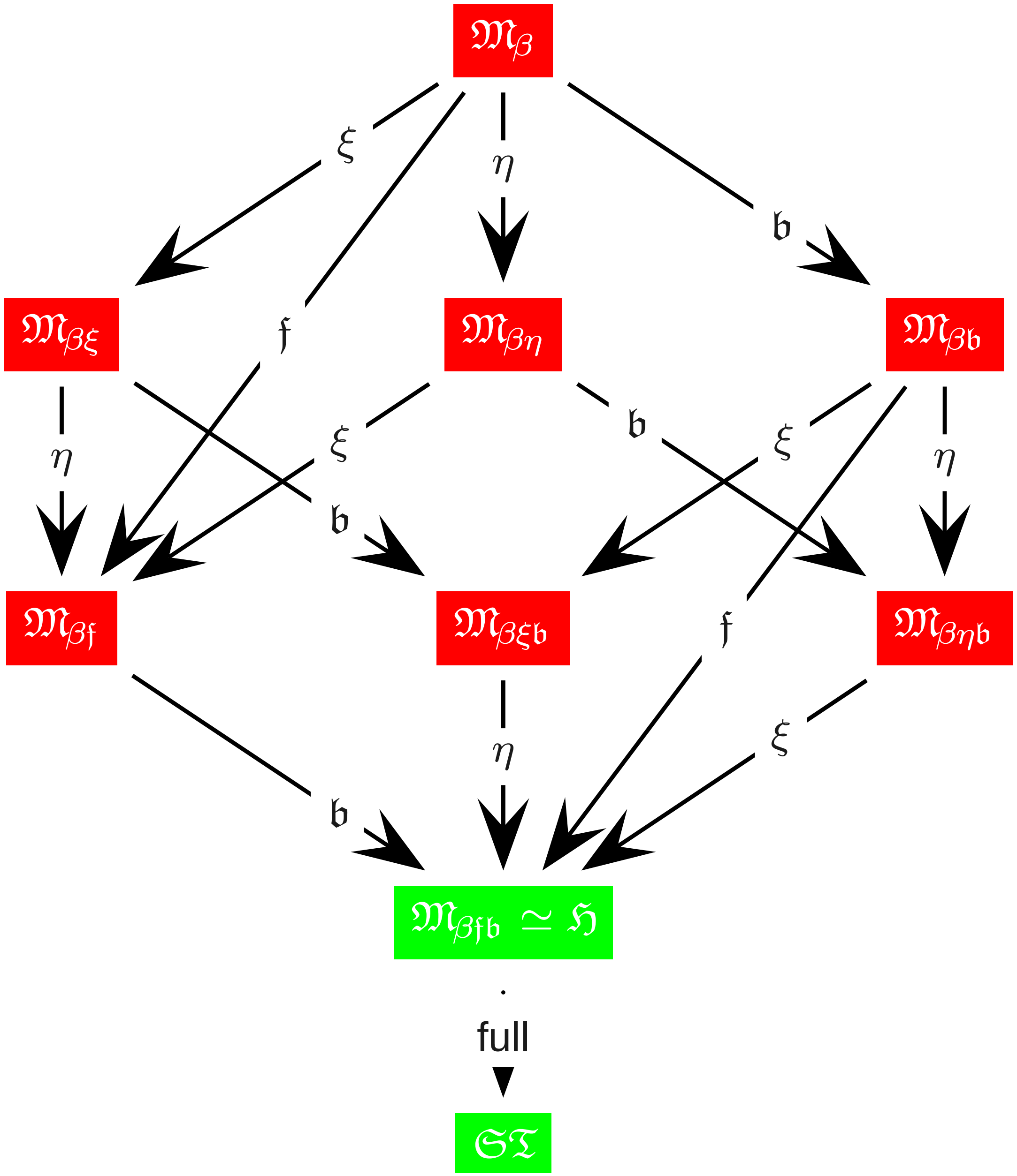


Playing with DeMorgan's Law:

- $\forall X, Y.\, X \land Y \Leftrightarrow \neg(\neg X \lor \neg Y)$
- $\forall X, Y.\, X \land Y \stackrel{*}{=} \neg(\neg X \lor \neg Y)$
- $(\lambda U \lambda V.\, U \land V) \stackrel{*}{=} (\lambda X \lambda Y.\, \neg(\neg X \lor \neg Y))$

requires $\mathfrak{b}$ and ξ

HOL-Problems: DeMorgan's Law



Playing with DeMorgan's Law:

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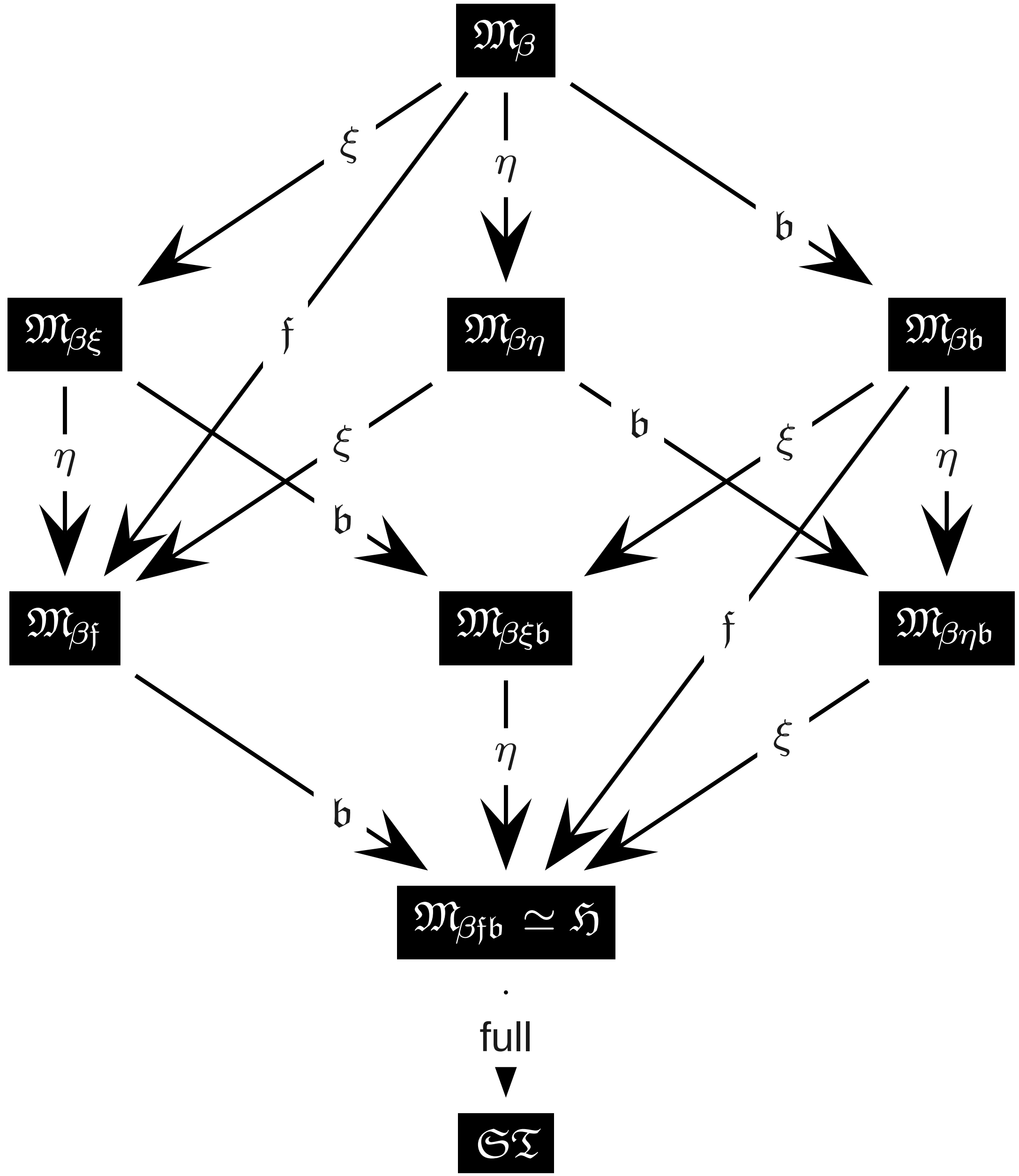
- $(\lambda U \lambda V.\, U \wedge V) \stackrel{*}{\doteq} (\lambda X \lambda Y.\, \neg(\neg X \vee \neg Y))$
- $\wedge \stackrel{*}{\doteq} (\lambda X \lambda Y.\, \neg(\neg X \vee \neg Y))$

requires $\mathfrak{b}$ and $\mathfrak{f}$

HOL-Problems: Set Comprehension



Set comprehension

- big challenge for automation
- [Benzm.BrownKohlhase-Draft-05] set instantiations can be used to simulate cut-rule if one of the following axioms is given: comprehension, induction, extensionality, choice, description
- dependend on logical constants in $\mathcal{S}$

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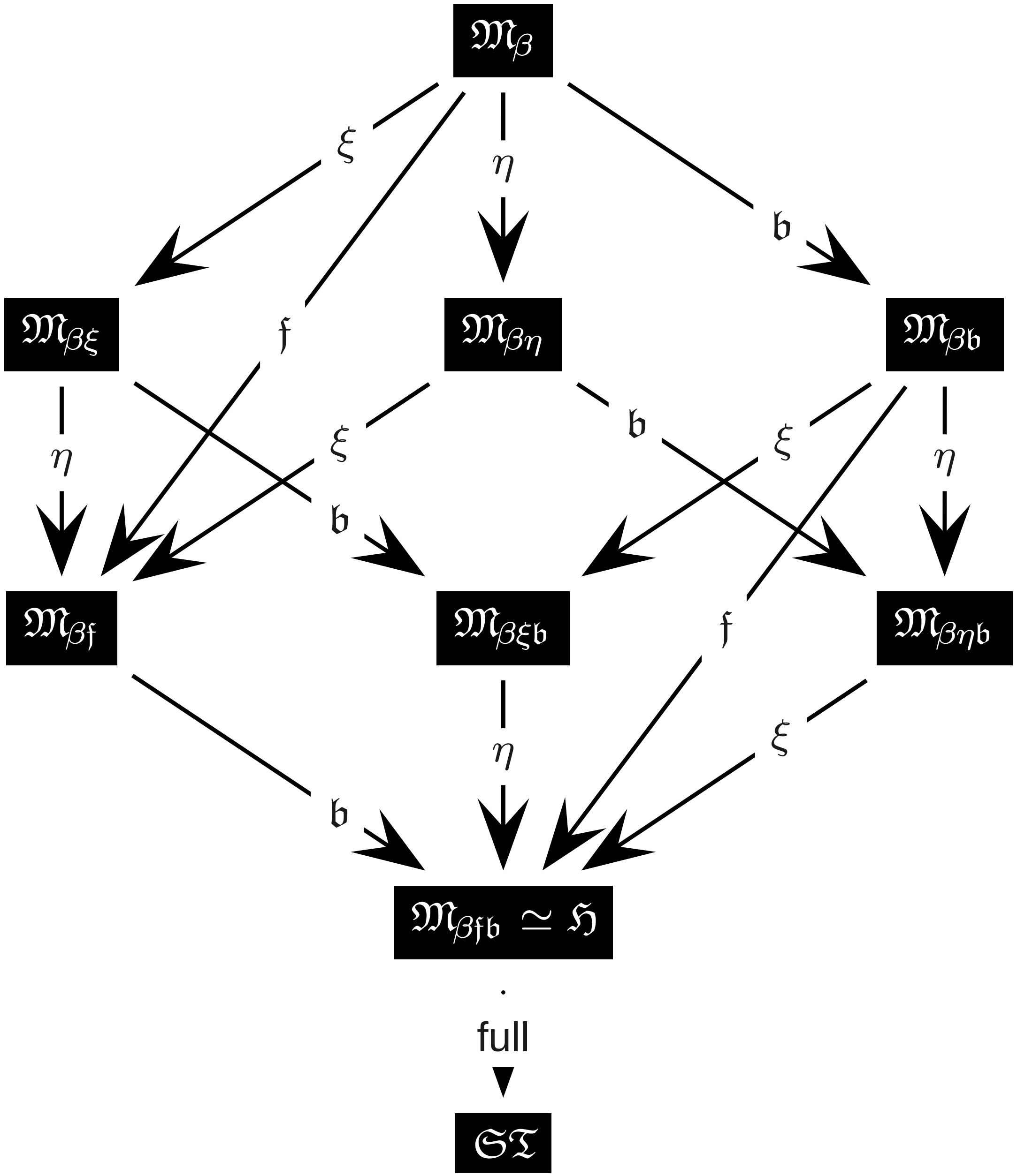

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- no property $\mathfrak{q}$ assumed

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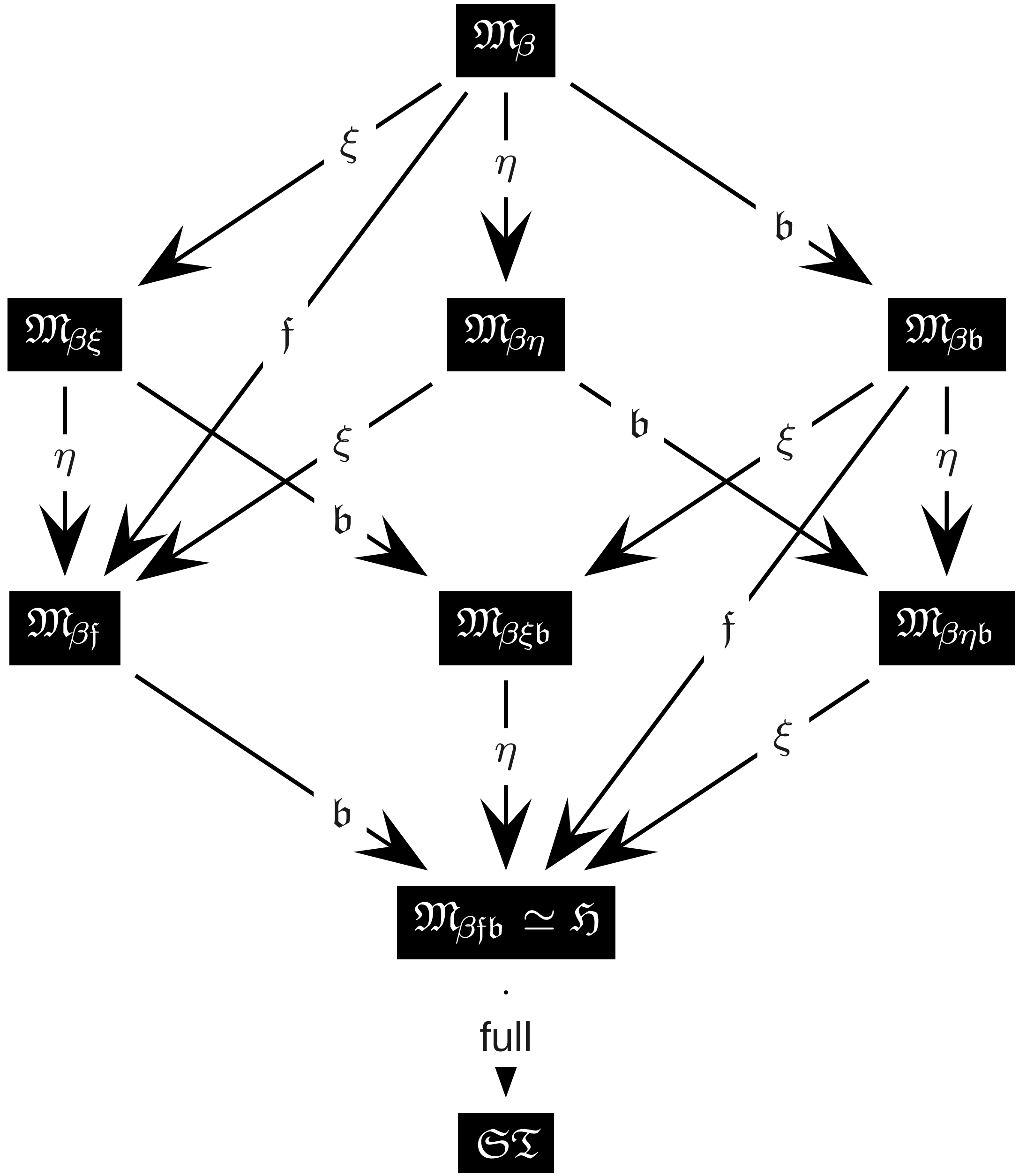

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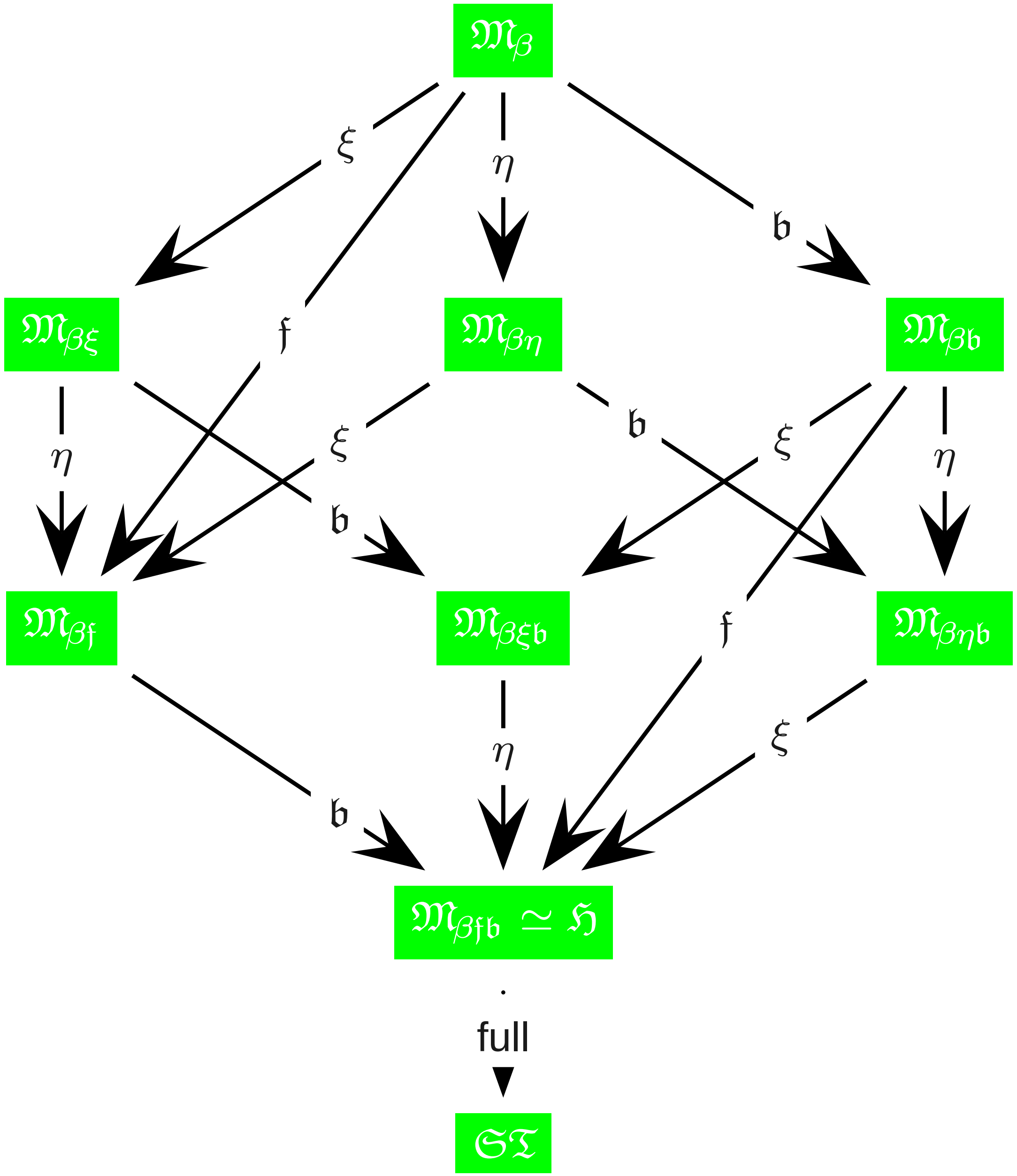
External vs. internal logical constants

- if $\neg \notin \mathcal{S}$:
  $\neg$ refers to 'external' symbol
  $\mathcal{M} \models \neg$A means $\mathcal{M} \not\models$ A

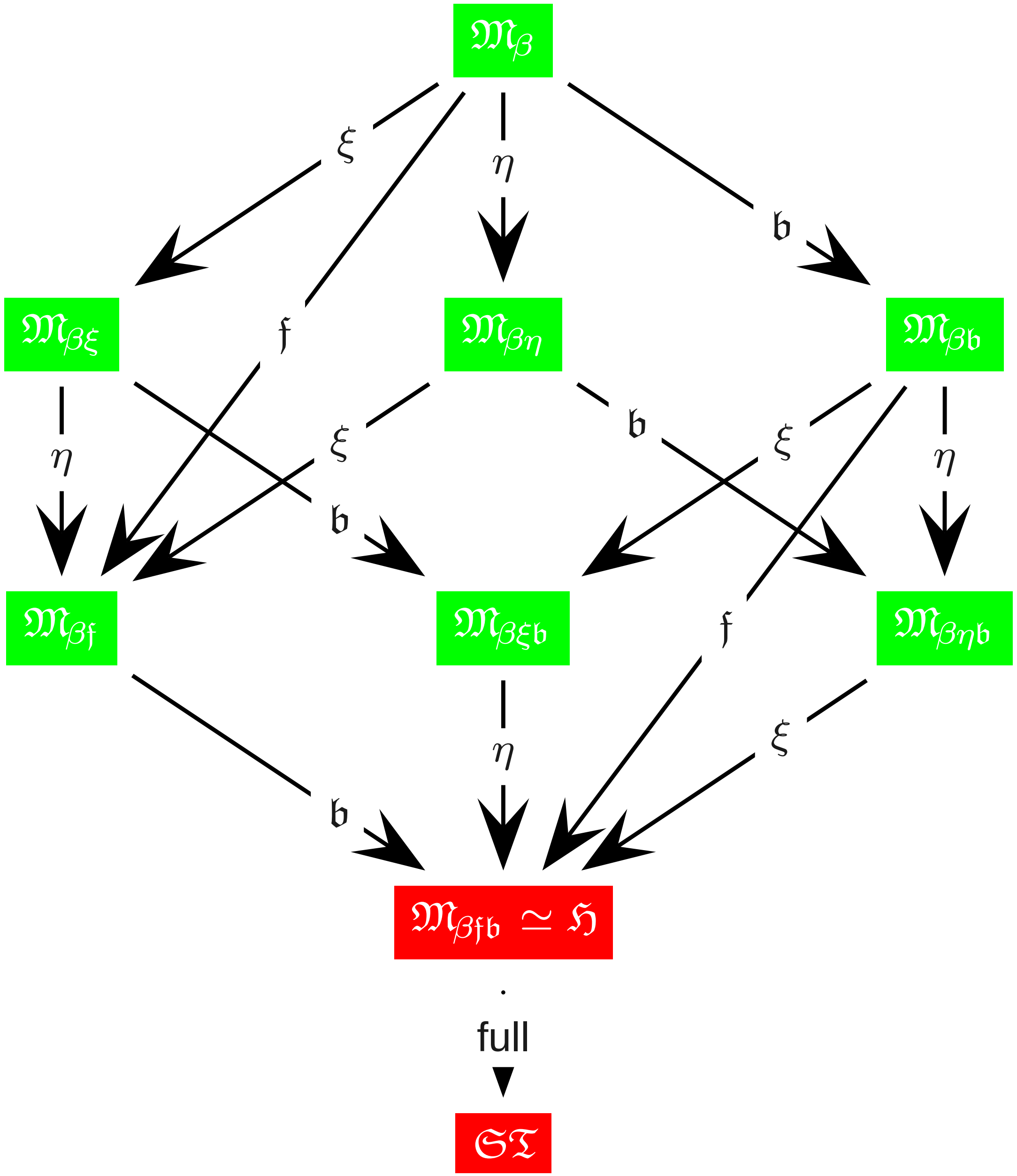
HOL-Problems: Set Comprehension



Set comprehension

- $\exists N_{oo} \forall P_o. \, NP \Leftrightarrow \neg P$
  - if $\neg \in \mathcal{S}$ or $\{\bot, \Rightarrow\} \subseteq \mathcal{S}$ or $\{\bot, \Leftrightarrow\} \subseteq \mathcal{S}$
  - e.g.: $N_{oo} \longleftarrow \lambda X_o. \, \neg X$
    e.g.: $N_{oo} \longleftarrow \lambda X_o. \, X \Rightarrow \bot$

HOL-Problems: Set Comprehension



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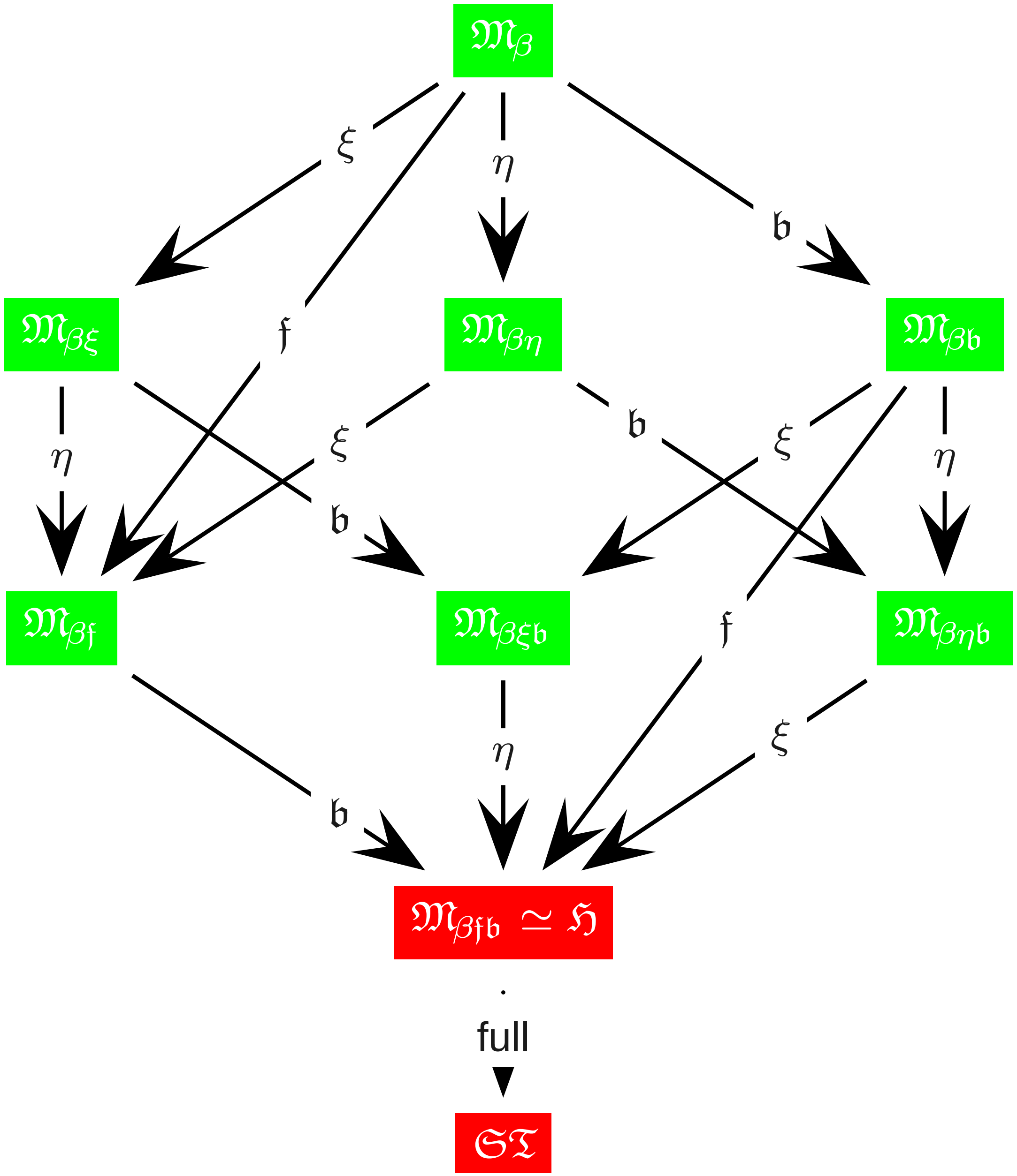

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Other examples from [Brown-PhD-04]

- Surjective Cantor Theorem
- Injective Cantor Theorem

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  - highlight some semantical or technical point

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  - Integrate them with proof assistants

Thank You

[LPAR-04] HOL versus FOL

SET171+3	$\forall X_{o\alpha}, Y_{o\alpha}, Z_{o\alpha}.X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
SET611+3	$\forall X_{o\alpha}, Y_{o\alpha}.(X \cap Y = \emptyset) \Leftrightarrow (X \setminus Y = X)$
SET624+3	$\forall X_{o\alpha}, Y_{o\alpha}, Z_{o\alpha}.\mathtt{Meets}(X, Y \cap Z) \Leftrightarrow \mathtt{Meets}(X, Y) \vee \mathtt{Meets}(X, Z)$
SET646+3	$\forall x_{\alpha}, y_{\beta}.\mathtt{Subrel}(\mathtt{Pair}(x, y), (\lambda u_{\alpha}.\top) \times (\lambda v_{\beta}.\top))$
SET670+3	$\forall Z_{o\alpha}, R_{o\beta\alpha}, X_{o\alpha}, Y_{o\beta}.\mathtt{IsRelOn}(R, X, Y) \Rightarrow \mathtt{IsRelOn}(\mathtt{RestrictRDom}(R, Z), Z, Y)$

$\_ \in \_$	:=	$\lambda x_{\alpha}, A_{o\alpha}.[Ax]$
$\emptyset$	:=	$[\lambda x_{\alpha}.\bot]$
$\_ \cap \_$	:=	$\lambda A_{o\alpha}, B_{o\alpha}.[\lambda x_{\alpha}.x \in A \wedge x \in B]$
$\_ \cup \_$	:=	$\lambda A_{o\alpha}, B_{o\alpha}.[\lambda x_{\alpha}.x \in A \vee x \in B]$
$\_ \setminus \_$	:=	$\lambda A_{o\alpha}, B_{o\alpha}.[\lambda x_{\alpha}.x \in A \vee x \notin B]$
$\mathtt{Meets}(\_, \_)$	:=	$\lambda A_{o\alpha}, B_{o\alpha}.[\exists x_{\alpha}.x \in A \wedge x \in B]$
$\mathtt{Pair}(\_, \_)$	:=	$\lambda x_{\alpha}, y_{\beta}.[\lambda u_{\alpha}, v_{\beta}.u = x \wedge v = y]$
$\_ \times \_$	:=	$\lambda A_{o\alpha}, B_{o\beta}.[\lambda u_{\alpha}, v_{\beta}.u \in A \wedge v \in B]$
$\mathtt{Subrel}(\_, \_)$	:=	$\lambda R_{o\beta\alpha}, Q_{o\beta\alpha}.[\forall x_{\alpha}, y_{\beta}.Rxy \Rightarrow Qxy]$
$\mathtt{IsRelOn}(\_, \_, \_)$	:=	$\lambda R_{o\beta\alpha}, A_{o\alpha}, B_{o\beta}.[\forall x_{\alpha}, y_{\beta}.Rxy \Rightarrow x \in A \wedge y \in B]$
$\mathtt{RestrictRDom}(\_, \_)$	:=	$\lambda R_{o\beta\alpha}, A_{o\alpha}.[\lambda x_{\alpha}, y_{\beta}.x \in A \wedge Rxy]$