



# Extensional Higher-Order Resolution Paramodulation and RUE-Resolution

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# Motivation

- Theorem Proving in Higher-Order Logic

- Examples:

1.  $p_{o \rightarrow o} a_o \wedge p_{o \rightarrow o} b_o \Rightarrow p_{o \rightarrow o} (a_o \wedge b_o)$

2.  $(\lambda X_\alpha. \text{young}_{\alpha \rightarrow o} X \wedge \text{male}_{\alpha \rightarrow o} X) = (\lambda X_\alpha. \text{male } X \wedge \text{young } X)$

3.  $\wp(\emptyset_{\gamma \rightarrow o}) = \{\emptyset\}$

Definitions:

$$\wp := \lambda A_{\gamma \rightarrow o}. \lambda B_{\gamma \rightarrow o}. \overbrace{\forall X_\gamma. B X \Rightarrow A X}^{B \subseteq A}$$

$$\emptyset_{\gamma \rightarrow o} := \lambda X_\gamma. \perp$$

$$\{\emptyset\}_{(\gamma \rightarrow o) \rightarrow o} := \lambda Y_{\gamma \rightarrow o}. (Y = (\lambda X_\gamma. \perp))$$

Expanded:

$$\neg((\lambda B_{\gamma \rightarrow o}. (\forall X_\gamma. (B X) \Rightarrow \perp)) = (\lambda Y_{\gamma \rightarrow o}. (Y = (\lambda X_\gamma. \perp))))$$

- Challenge: Equality and Extensionality / avoid additional axioms



# Overview

- Type Theory: Syntax and Semantics
- Leibniz Equality, Extensionality
- Extensional Higher-Order Resolution
- Primitive Equality
  - Extensional Higher-Order Paramodulation
  - Extensional Higher-Order RUE-Resolution
- (Completeness: Adaption of the Abstract Consistency Method)
- Implementation and Examples



# Type Theory: HOL & Simply Typed $\lambda$ -Calc.

- Types: (i)  $\{i, o\} \in T$  (ii)  $\alpha, \beta \in T$ , then  $\alpha \rightarrow \beta \in T$

- Terms:

(i)  $V_\alpha \subseteq \Lambda$ ;  $V_\alpha$  (infinite) Sets of Variables ( $\alpha \in T$ )

(ii)  $C_\alpha \subseteq \Lambda$ ;  $C_\alpha$  Sets of Constants ( $\alpha \in T$ )

Required:  $\neg_{o \rightarrow o} \in C_{o \rightarrow o}$ ,  $\forall_{o \rightarrow (o \rightarrow o)} \in C_{o \rightarrow (o \rightarrow o)}$ ,

$\Pi_{(\alpha \rightarrow o) \rightarrow o} \in C_{(\alpha \rightarrow o) \rightarrow o}$  ( $\alpha \in T$ )

(iii) Application:  $\mathbf{A}_{\alpha \rightarrow \beta}, \mathbf{B}_\alpha \in \Lambda$ , then  $(\mathbf{A} \mathbf{B})_\beta \in \Lambda$

(iii) Abstraction:  $X_\alpha \in V_\alpha, \mathbf{A}_\beta \in \Lambda$ , then  $(\lambda X. \mathbf{A})_{\alpha \rightarrow \beta} \in \Lambda$

- $\lambda$ -Conversion /  $\beta$ -Normalform /  $\beta\eta$ -(Head-)Normalform:

$$\lambda X_\gamma. \mathbf{A} \leftrightarrow^\alpha \lambda Y_\gamma. \mathbf{A}[Y/X]$$

$$(\lambda X_\gamma. \mathbf{A}) \mathbf{B}_\gamma \rightarrow^\beta \mathbf{A}[\mathbf{B}/X] \quad \lambda X. \mathbf{A} X \rightarrow^\eta \mathbf{A} \text{ if } X \notin \text{Free}(\mathbf{A})$$



# Standard Semantics

- Domains: Choose:  $D_\iota$   
Required:  $D_o = \{\perp, \top\}$ ,  $D_{\alpha \rightarrow \beta} = \text{Funcs}(D_\alpha, D_\beta)$
- Interpretation: Choose:  $I_\alpha : C_\alpha \longrightarrow D_\alpha$   
Required:  $I(\neg_{o \rightarrow o})$  and  $I(\forall_{o \rightarrow (o \rightarrow o)})$  as intended

$I(\Pi_{(\alpha \rightarrow o) \rightarrow o})$  is a predicate  $p \in D_{(\alpha \rightarrow o) \rightarrow o}$ , such that for every  $q_{\alpha \rightarrow o} \in D_{\alpha \rightarrow o} : p q_{\alpha \rightarrow o} = \top$  iff  $q$  holds for all  $a \in D_\alpha$   
 $\Rightarrow \forall X_\alpha. \mathbf{A}_o$  is coded as  $\Pi (\lambda X_\alpha. \mathbf{A}_o)$

- Variable Assignment:  $\varphi_\alpha : V_\alpha \longrightarrow D_\alpha$
- Interpretation of terms:  $I_\varphi : \Lambda_\alpha \longrightarrow D_\alpha$   
 $I_\varphi(X) = \varphi(X)$ ,  $I_\varphi(c) = I(c)$ ,  $I_\varphi(\mathbf{A} \mathbf{B}) = I_\varphi(\mathbf{A}) I_\varphi(\mathbf{B})$ ,  
 $I_\varphi(\lambda X_\alpha. \mathbf{B}_\beta) = f \in D_{\alpha \rightarrow \beta}$ , such that  $fa = I_{\varphi[a/X]}(\mathbf{B})$  for all  $a \in D_\alpha$
- Model:  $\mathcal{M} = (\mathcal{D} : \{D_\alpha\}, \mathcal{I} : \{I_\alpha\})$ , Satisfiability and Validity as usual

# Henkin Semantics

- like Standard Semantics but:  $D_{\alpha \rightarrow \beta} \subseteq Functions(D_\alpha, D_\beta)$
  - Required:  $I_\Phi$  is total (i.e. each term has a Denotation)
  - It holds:
    - Each Standard Model is a Henkin Model
    - There are more Henkin Models as Standard Models
    - A formula that is valid in Henkin Sem. is also valid in Standard Sem.
    - There are less formulae valid in Henkin semantics
- ⇒ Goedel 1931: Standard Semantics allows no complete calculi
- ⇒ Henkin 1950: Most general notion that allows complete calculi



# Properties of Type Theory

- Comprehension Principles are built-in ( $\exists F_{\alpha \rightarrow \beta} \cdot \forall X_{\alpha} \cdot (F X) = \mathbf{A}_{\beta}$ )
- Optional: Axiom of Choice ( $\exists F_{(\alpha \rightarrow o) \rightarrow \alpha} \cdot \forall M_{\alpha \rightarrow o} \cdot (\exists X_{\alpha} \cdot M X) \Rightarrow M (F M)$ ) and Descriptionoperator  $\iota$
- Leibniz Equality denotes intended Equalityrelation (i.e. a functional congruencerelation)

$$\doteq^{\alpha} := \lambda X_{\alpha} \cdot \lambda Y_{\alpha} \cdot \forall P_{\alpha \rightarrow o} \cdot P X \Rightarrow P Y$$

$$\text{i.e. : } a_{\alpha} \doteq^{\alpha} b_{\alpha} \quad \text{expands to} \quad \forall P_{\alpha \rightarrow o} \cdot P a \Rightarrow P b$$

$\Rightarrow$  Equality is built-in in Type Theory (with Standard or Henkin Semantics)

but ...



# Disadvantages of Leibniz Equality

- Extensionality Axioms needed: E.g. Andrews' Higher Order Resolution (1971), Huet's Constrained Resolution (1972), Jensen & Pietrowski (1972), Wolfram (1993), Kohlhase (1994), TPS-System, HOL-System

– EXT-Func<sup>≐</sup> :  $\forall F_{\alpha \rightarrow \beta} \cdot \forall G_{\alpha \rightarrow \beta} (\forall X_{\beta} \cdot F X \doteq G X) \Rightarrow F \doteq G$

expanded:  $\forall F_{\alpha \rightarrow \beta} \cdot \forall G_{\alpha \rightarrow \beta} \cdot (\forall X_{\beta} \cdot \forall P_{\beta \rightarrow o} \cdot P (F X) \Rightarrow P (G X) \Rightarrow \forall Q_{(\alpha \rightarrow \beta) \rightarrow o} \cdot P F \Rightarrow P G$

clauses:  $\mathcal{C}_1 : [p_{\beta \rightarrow o} (F s_{\beta})]^T \vee [Q F]^F \vee [Q G]^T, \mathcal{C}_2 : [p_{\beta \rightarrow o} (G s_{\beta})]^T \vee [Q F]^F \vee [Q G]^T$

– EXT-Bool<sup>≐</sup> :  $\forall A_o \cdot \forall B_o \cdot (A \Leftrightarrow B) \Leftrightarrow A \doteq^o B$

expanded:  $\forall A_o \cdot \forall B_o \cdot (A \Leftrightarrow B) \Leftrightarrow (\forall Q_{o \rightarrow o} \cdot Q A \Rightarrow Q B$

clauses:  $\mathcal{C}_1 : [A]^F \vee [B]^F \vee [P A]^F \vee [P B]^T, \mathcal{C}_2 : [A]^T \vee [B]^T \vee [P A]^F \vee [P B]^T, \mathcal{C}_3 :$

$[A]^F \vee [B]^T \vee [p A]^T, \mathcal{C}_4 : [A]^F \vee [B]^T \vee [p B]^F, \mathcal{C}_5 : [A]^T \vee [B]^F \vee [p A]^T, \mathcal{C}_6 : [A]^T \vee [B]^F \vee [p B]^F$



# Extensional HO Resolution $\mathcal{ER}$ I

Clause Normalisation  $\mathcal{CNF}$

$$\frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^T}{\mathbf{C} \vee [\mathbf{A}]^T \vee [\mathbf{B}]^T} \vee^T \quad \frac{\mathbf{C} \vee [\mathbf{A} \wedge \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{A}]^F} \vee_l^F \quad \frac{\mathbf{C} \vee [\mathbf{A} \wedge \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{B}]^F} \vee_r^F$$

$$\frac{\mathbf{C} \vee [\neg \mathbf{A}]^T}{\mathbf{C} \vee [\mathbf{A}]^F} \neg^T \quad \frac{\mathbf{C} \vee [\neg \mathbf{A}]^F}{\mathbf{C} \vee [\mathbf{A}]^T} \neg^F \quad \frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^T \quad X_\alpha \text{ new variable}}{\mathbf{C} \vee [\mathbf{A} X]^T} \Pi^T$$

$$\frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^F \quad s_\alpha \text{ is a Skolem term for this clause}}{\mathbf{C} \vee [\mathbf{A} s_\alpha]^F} \Pi^F$$



# Extensional HO Resolution $\mathcal{ER}$ II

Constrained Resolution

$$\frac{[A]^\alpha \vee C \quad [B]^\beta \vee D \quad \alpha \neq \beta}{C \vee D \vee [A = B]^F} \text{Res} \qquad \frac{[A]^\alpha \vee [B]^\alpha \vee C \quad \alpha \in \{T, F\}}{[A]^\alpha \vee C \vee [A = B]^F} \text{Fac}$$

$$\frac{[Q_\gamma \overline{U}^k]^\alpha \vee C \quad P \in \mathcal{GB}_\gamma^{\{\neg, \vee\} \cup \{\Pi^\beta \mid \beta \in \mathcal{T}^k\}}}{[Q_\gamma \overline{U}^k]^\alpha \vee C \vee [Q = P]^F} \text{Prim}$$

Note: Resolution or Factorization on Unification Constraints is not allowed

Primitive Substitution:  $\exists P_{\alpha \rightarrow o} \cdot P \ a_\alpha \xrightarrow{CNF} C_1 : [P \ a]^F$

$$\text{Prim}(C_1, [\lambda X_\alpha \cdot \neg(P' X) / P]) : C_2 : [P' a]^T$$



# Extensional HO Resolution $\mathcal{ER}$ III

Higher-Order Pre-Unification

$$\frac{\mathbf{C} \vee [\mathbf{A}_{\alpha \rightarrow \beta} \ \mathbf{C}_{\alpha} = \mathbf{B}_{\alpha \rightarrow \beta} \ \mathbf{D}_{\alpha}]^F}{\mathbf{C} \vee [\mathbf{A} = \mathbf{B}]^F \vee [\mathbf{C} = \mathbf{D}]^F} \text{Dec}$$

$$\frac{\mathbf{C} \vee [\mathbf{A} = \mathbf{A}]^F}{\mathbf{C}} \text{Triv}$$

$$\frac{\mathbf{C} \vee [F_{\gamma} \ \overline{\mathbf{U}}^n = h \ \overline{\mathbf{V}}^m]^F \quad \mathbf{G} \in \mathcal{GB}_{\gamma}^h}{\mathbf{C} \vee [F = \mathbf{G}]^F \vee [F \ \overline{\mathbf{U}}^n = h \ \overline{\mathbf{V}}^m]^F} \text{FlexRigid}$$

$$\frac{\mathbf{C} \vee [(\lambda X_{\alpha} \cdot \mathbf{M}_{\beta}) = \mathbf{N}_{\alpha \rightarrow \beta}]^F}{\mathbf{C} \vee [(\lambda X_{\alpha} \cdot \mathbf{M}) \ s = \mathbf{N} \ s]^F} \text{Func}_1 \quad \frac{\mathbf{C} \vee [(\lambda X_{\alpha} \cdot \mathbf{M}_{\beta}) = (\lambda Y_{\alpha} \cdot \mathbf{N}_{\beta})]^F}{\mathbf{C} \vee [(\lambda X_{\alpha} \cdot \mathbf{M}) \ s = (\lambda X_{\alpha} \cdot \mathbf{N}) \ s]^F} \text{Func}_2$$

$$\frac{\mathbf{C} \vee [X = \mathbf{A}]^F \quad X \notin \text{Free}(\mathbf{A})}{(\mathbf{C}[\mathbf{A}/X])} \text{Subst}$$

$$\frac{\mathcal{D} \quad \mathcal{C} \in \mathcal{CNF}(\mathcal{D})}{\mathcal{C}} \text{Cnf}$$



# Extensional HO Resolution $\mathcal{ER}$ IV

Extensionality: Recursive calls of the overall refutation process from within the unification process

$$\frac{C \vee [M_o = N_o]^F}{C \vee [M_o \leftrightarrow N_o]^F} \textit{Equiv} \qquad \frac{C \vee [M_\alpha = N_\alpha]^F}{C \vee [\forall P_{\alpha \rightarrow o}. P M \Rightarrow P N]^F} \textit{Leib}$$

$$\frac{C \vee [M_{\alpha \rightarrow \beta} = N_{\alpha \rightarrow \beta}]^F \quad s_\alpha \text{ Skolem term for this clause}}{C \vee [M_s = N_s]^F} \textit{Func}$$

- Claim: Rule Leib can be restricted to type  $\iota$
- $\Rightarrow$  First Henkin complete refutation calculus that does not need additional axiom in the search space (CADE-15)
- $\Rightarrow$  Difference to Huet (1972): Eager Unification becomes essential



# Example: $(\lambda X_{\alpha} \blacksquare \text{joung}_{\alpha \rightarrow o} X \wedge \text{male}_{\alpha \rightarrow o} X) = (\lambda X_{\alpha} \blacksquare \text{male } X \wedge \text{joung } X)$

$$\forall P_{(\alpha \rightarrow o) \rightarrow o} \blacksquare P (\lambda X_{\alpha} \blacksquare \text{joung}_{\alpha \rightarrow o} X \wedge \text{male}_{\alpha \rightarrow o} X) \Rightarrow P (\lambda X_{\alpha} \blacksquare \text{male } X \wedge \text{joung } X)$$

$$c1: [p (\lambda X \blacksquare \text{joung } X \wedge \text{male } X)]^T$$

$$c2: [p (\lambda X \blacksquare \text{male } X \wedge \text{joung } X)]^F$$

$$c3: [(p (\lambda X \blacksquare \text{joung } X \wedge \text{male } X)) = (p (\lambda X \blacksquare \text{male } X \wedge \text{joung } X))]^F$$

$$c4: [(\lambda X \blacksquare \text{joung } X \wedge \text{male } X) = (\lambda X \blacksquare \text{male } X \wedge \text{joung } X)]^F$$

$$c5: [(\text{joung } s \wedge \text{male } s) = (\text{male } s \wedge \text{joung } s)]^F$$

$$c6: [(\text{joung } s \wedge \text{male } s) \equiv (\text{male } s \wedge \text{joung } s)]^F$$

$$c7: [\text{joung } s]^T \vee [\text{male } s]^T$$

$$c8: [\text{joung } s]^T$$

$$c9: [\text{male } s]^T$$

$$c10: [\text{joung } s]^F \vee [\text{male } s]^F$$



# Adding Primitive Equality

- Why: Leibniz Equality introduces flexible heads (Primitive Substitution)

⇒ Improve Equality Treatment by adding a Primitive Equality Treatment

- Reflexivity Definition:

$$\doteq^\alpha := \lambda X_\alpha. \lambda Y_\alpha. \forall Q_{\alpha \rightarrow \alpha \rightarrow o}. (\forall Z_\alpha. (Q Z Z)) \Rightarrow (Q X Y)$$

- Modified Leibniz Equality:

$$\doteq^\alpha := \lambda X_\alpha. \lambda Y_\alpha. \forall P_{\alpha \rightarrow o}. ((a_o \vee \neg a_o) \wedge P X) \Rightarrow ((b_o \vee \neg b_o) \wedge P Y)$$

⇒ It is not decidable whether a given input problem contains Defined Equations

⇒ We still have to take care of Defined Equations even if we now add a Primitive Equality Treatment to the calculus



# Extensional HO Paramodulation $\mathbb{P}$ I

$$\frac{[\mathbf{A}[\mathbf{T}_\beta]]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[\mathbf{A}[\mathbf{R}]]^\alpha \vee C \vee D \vee [\mathbf{T} =^\beta \mathbf{L}]^F} \text{ Para}$$

$$\frac{[\mathbf{A}]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[P_{\alpha \rightarrow o} \mathbf{R}]^\alpha \vee C \vee D \vee [\mathbf{A} =^o P_{\beta \rightarrow o} \mathbf{L}]^F} \text{ Para}'$$

- negative Equation Literals are still interpreted as Unification Constraints
- No Resolution on Unification Constraints

$$\frac{[p(f(f a))]^T \quad [f = g]^T}{[p(f(g a))]^T} \text{ Para, Uni}$$

$$\frac{[p(f(f a))]^T \quad [f = g]^T}{[p(g(f a))]^T} \text{ Para, Uni}$$

$$\frac{[p(f(f a))]^T \quad [f = g]^T}{[P g]^T \vee [(P f) = (p(f(f a)))]^F} \text{ Para}'$$

$$\frac{[p(f(f a))]^T \text{ with } [\lambda X \bullet (p(f(f a)))/P]}{[p(f(g a))]^T \text{ with } [\lambda X \bullet (p(f(X a)))/P]} \text{ UNI}$$

$$\frac{[p(f(g a))]^T \text{ with } [\lambda X \bullet (p(f(X a)))/P]}{[p(g(f a))]^T \text{ with } [\lambda X \bullet (p(X(f a)))/P]} \text{ UNI}$$

$$\frac{[p(g(f a))]^T \text{ with } [\lambda X \bullet (p(X(f a)))/P]}{[p(g(g a))]^T \text{ with } [\lambda X \bullet (p(X(X a)))/P]} \text{ UNI}$$



# Extensional HO Paramodulation EP II

- In FO Reflexivity Rule needed  $\longrightarrow$  already given here by UNI

$$\frac{[(fX) = (fa)]^F}{\square} \text{Ref}$$

$$\frac{[(fX) = (fa)]^F}{\square} \text{UNI}$$

- $C_1 : [\underbrace{\text{empty}(\lambda X. \text{young } X \wedge \text{male } X \wedge \text{smart } X))}_{\{X | \text{young } X \wedge \text{male } X \wedge \text{smart } X\} \in \text{empty}}]^T$       $C_2 : [\underbrace{\forall X. (\text{young } X \wedge \text{male } X) = \text{boy } X}_{\forall X. (\text{young } X \wedge \text{male } X) = \text{boy } X}]^T$   
 To show:  $C_3 : [\underbrace{\text{empty}(\lambda X. \text{boy } X \wedge \text{smart } X)}_{\{X | \text{boy } X \wedge \text{smart } X\} \in \text{empty}}]^F$

- Employ Term-Rewriting Idea with Paramodulation rule:

$$\text{Para}(C_1, C_2), \text{UNI} : C_4 : [\text{empty}(\lambda X. \text{boy } X \wedge \text{smart } X)]^T$$

$$\text{Res}(C_4, C_3), \text{UNI} : \square$$



# New Problem in $\mathcal{EP}$ : Positive Equation Literals

- Contrast to FO: Contradictory positive Equation Literals

$$[\mathbf{A}_o = \neg \mathbf{A}_o]^T \quad \overbrace{[(\lambda X. male X) = (\lambda X. \neg(male X))]}^{\{X | male X\} = \overline{\{X | male X\}}}]^T$$

- Additional positive Extensionality Rules needed

$$\frac{\mathbf{C} \vee [\mathbf{M}_o = \mathbf{N}_o]^T}{\mathbf{C} \vee [\mathbf{M}_o \Leftrightarrow \mathbf{N}_o]^T} \textit{Equiv}' \quad \frac{\mathbf{C} \vee [\mathbf{M}_{\alpha \rightarrow \beta} = \mathbf{N}_{\alpha \rightarrow \beta}]^T \quad X \textit{ new}}{\mathbf{C} \vee [\mathbf{M} X = \mathbf{N} X]^T} \textit{Func}'$$

- New Rules further strengthen the Difference-Reduction character
- Henkin Completeness without additional axioms  
(proved yet only with additional FlexFlex-Rule in UNI)



# Extensional HO Paramodulation EP III

- Slightly modify the problem: No Term-Rewriting possible at all !

$$\mathcal{C}'_1 : [nempty(\lambda X_\iota. \blacksquare((joun\text{g } X \wedge smart\ X) \wedge male\ X))]^T \quad \mathcal{C}_2 : [(joun\text{g } X \wedge male\ X) = boy\ X]^T$$

$$\text{To show: } \mathcal{C}_3 : [nempty(\lambda X_\iota. \blacksquare boy\ X \wedge smart\ X)]^F$$

$$Para(\mathcal{C}_1, \mathcal{C}_2) : \quad \mathcal{C}_4 : [nempty(\lambda X. \blacksquare (boy\ X \wedge smart\ X))]^T \vee [(joun\text{g } X \wedge male\ X) = (joun\text{g } X \wedge smart\ X)]^F$$

...

Instead one has to employ the Difference-Reduction idea

$$Res(\mathcal{C}'_1, \mathcal{C}_3) : \quad \mathcal{C}_4 : [(nempty(\lambda X_\iota. \blacksquare((joun\text{g } X \wedge smart\ X) \wedge male\ X))) = (nempty(\lambda X_\iota. \blacksquare boy\ X \wedge smart\ X))]^F$$

$$Dec(\mathcal{C}_4), Func, Equiv : \quad \mathcal{C}_5 : [((joun\text{g } s \wedge smart\ s) \wedge male\ s) \equiv (boy\ s \wedge smart\ s)]^F$$

$$Equiv'(\mathcal{C}_2) : \quad \mathcal{C}_6 : [(joun\text{g } X \wedge male\ X) \equiv boy\ X]^T$$

$$CNF(\mathcal{C}_5, \mathcal{C}_6) : \quad \dots$$



⇒ Unavoidable Mix of Term-Rewriting & Difference-Reduction



# Extensional HO RUE-Resolution ~~ERUE~~ I

- Motivation: Try to find a pure Difference-Reducing calculus
- Extensional HO RUE-Resolution:
  - Remove Paramodulation Rule and instead ...
  - Allow to resolve and factorize also on Unification Constraints

•  $\mathcal{C}_1 : [nempty (\lambda X_l. \blacksquare ((joun\ g\ X \wedge male\ X) \wedge smart\ X))]^T$        $\mathcal{C}_2 : [(joun\ g\ X \wedge male\ X) = boy\ X]^T$

$\mathcal{C}_3 : [nempty (\lambda X_l. \blacksquare boy\ X \wedge smart\ X)]^F$

$Res(\mathcal{C}_1, \mathcal{C}_3), Dec, Func :$        $\mathcal{C}_4 : [((joun\ g\ s \wedge male\ s) \wedge smart\ s) = (boy\ s \wedge smart\ s)]^F$

$Dec(\mathcal{C}_4), Triv :$        $\mathcal{C}_5 : [(joun\ g\ s \wedge male\ s) = boy\ s]^F$

$Res(\mathcal{C}_5, \mathcal{C}_2), UNI :$       ...  $\square$



# Extensional HO RUE-Resolution ~~ERUE~~ II

- Slightly modified example

- $\mathcal{C}_1 : [nempty (\lambda X_t. ((jounq X \wedge smart X) \wedge male X))]^T$        $\mathcal{C}_2 : [(jounq X \wedge male X) = boy X]^T$

$$\mathcal{C}_3 : [nempty (\lambda X_t. boy X \wedge smart X)]^F$$

$$Res(\mathcal{C}_1, \mathcal{C}_3), Dec, Func : \mathcal{C}_4 : [((jounq s \wedge smart s) \wedge male s) = (boy s \wedge smart s)]^F$$

$$Equiv(\mathcal{C}_4) : \mathcal{C}_5 : [((jounq s \wedge smart s) \wedge male s) \equiv (boy s \wedge smart s)]^F$$

$$Equiv'(\mathcal{C}_2) : \mathcal{C}_6 : [(jounq X \wedge male X) \equiv boy X]^T$$

$$CNF(\mathcal{C}_5, \mathcal{C}_6) : \dots \quad \square$$

⇒ Pure Difference-Reduction approach probably better to handle in practice

- Henkin Completeness without additional axioms  
(proved yet only with additional FlexFlex-Rule in UNI)



# Conclusion

- Henkin complete refutation approaches for classical Type Theory that do not need additional axiom and which treat equality and extensionality in a rather appropriate way:
  - Extensional HO Resolution  $\mathbb{ER}$
  - Extensional HO Paramodulation  $\mathbb{EP}$  (Compl. modulo FlexFlex)
  - Extensional HO RUE-Resolution  $\mathbb{ERUE}$  (Compl. modulo FlexFlex)
- Implementation LEO
- Experiments very promising for examples about sets
- For Completeness Proofs: Adaption of Smullyan's / Andrews' Unifying Principle to HOL with Henkin Semantics



# Further Work

- Clarify the many open questions
- Compare  $\text{EP}$ ,  $\text{EP}$  and  $\text{ERUE}$
- Integration of LEO as powerful Deductive Agent in  $\Omega\text{MEGA}$ 's Agentmechanism for supporting Interactive Theorem Proving (talk of last week)
- Cooperation with TPS and First-Order ATP's via Proof Planning layer in  $\Omega\text{MEGA}$
- Reimplementation of LEO in Oz: Exploit Concurrency



# Higher-Order Abstract Consistency

**Definition 0.1 (Properties for Abstract Consistency Classes).** Let  $\mathcal{I}_\Sigma$  be a class of sets of  $\Sigma$ -sentences.

- $\nabla_c$  If  $\mathbf{A}$  is atomic, then  $\mathbf{A} \notin \Phi$  or  $\neg\mathbf{A} \notin \Phi$ .
- $\nabla_{\neg}$  If  $\neg\neg\mathbf{A} \in \Phi$ , then  $\Phi * \mathbf{A} \in \mathcal{I}_\Sigma$ .
- $\nabla_\beta$  If  $\mathbf{A} \in \Phi$  and  $\mathbf{B}$  is the  $\beta$ -normal form of  $\mathbf{A}$ , then  $\mathbf{B} * \Phi \in \mathcal{I}_\Sigma$ .
- $\nabla_f$  If  $\mathbf{A} \in \Phi$  and  $\mathbf{B}$  is the  $\beta\eta$ -normal form of  $\mathbf{A}$ , then  $\mathbf{B} * \Phi \in \mathcal{I}_\Sigma$ .
- $\nabla_{\vee}$  If  $\mathbf{A} \vee \mathbf{B} \in \Phi$ , then  $\Phi * \mathbf{A} \in \mathcal{I}_\Sigma$  or  $\Phi * \mathbf{B} \in \mathcal{I}_\Sigma$ .
- $\nabla_{\wedge}$  If  $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$ , then  $\Phi \cup \{\neg\mathbf{A}, \neg\mathbf{B}\} \in \mathcal{I}_\Sigma$ .
- $\nabla_{\forall}$  If  $\Pi^\alpha \mathbf{F} \in \Phi$ , then  $\Phi * \mathbf{F}\mathbf{W} \in \mathcal{I}_\Sigma$  for each  $\mathbf{W} \in \text{cwff}_\alpha(\Sigma)$ .
- $\nabla_{\exists}$  If  $\neg\Pi^\alpha \mathbf{F} \in \Phi$ , then  $\Phi * \neg(\mathbf{F}w) \in \mathcal{I}_\Sigma$  for any constant  $w \in \Sigma_\alpha$ , which does not occur in  $\Phi$ .
- $\nabla_b$  If  $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$ , then  $\Phi \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \mathcal{I}_\Sigma$  or  $\Phi \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \mathcal{I}_\Sigma$ .
- $\nabla_q$  If  $\neg(\mathbf{F} \doteq^{\alpha \rightarrow \beta} \mathbf{G}) \in \Phi$ , then  $\Phi * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w) \in \mathcal{I}_\Sigma$  for any constant  $w \in \Sigma_\alpha$ , which does not occur in  $\Phi$ .
- $\nabla_e$  (r)  $\neg(\mathbf{A} =^\alpha \mathbf{A}) \notin \Phi$   
(s) if  $\mathbf{F}[\mathbf{A}]_p \in \Phi$  and  $\mathbf{A} = \mathbf{B} \in \Phi$ , then  $\Phi * \mathbf{F}[\mathbf{B}]_p \in \mathcal{I}_\Sigma$



# Higher-Order Abstract Consistency

